MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XXVI.

MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF THE GENERAL MEDICAL COUNCIL.

VOL. XXVI.

FROM JULY TO DECEMBER, 1876.

LONDON:
C. F. HODGSON & SON, GOUGH SQUARE,
FLEET STREET.

1876.

LONDON:

PRINTED BY C. F. HODGSON AND SON,

GOUGH SQUARE, FLEET STREET.

LIST OF CONTRIBUTORS.

ALDIS, J. S., M.A.; H.M. Inspector of Schools.
ALLMAN, GEO. S., LL.D.; Professor of Mathematics in the Queen's University, Galway.
ARMENANTE, Professor, Pesaro.
BALL, ROBT. STAWELL, LL.D.; Astronomer Royal of Ireland, and Prof. of Astronomy in the University of Dublin.
BATES, GILL; Fenis, Illinois, United States.
BATTAGLINI, GIUSEPPE; Professore di Matematiche nell' Università di Roma.
BATTLE, RICHARD; Stafford.
BELTRAMI, Professor E.; University of Pisa.
BIRCH, Rev. J. G., M.A.; R. M. A., Woolwich.
BLACKWOOD, ELIZABETH; New York.
BLISSARD, Rev. J., B.A.; The Vicarage, Hampstead Norris, Berks.
BOOTH, Dr., F.R.S.; The Vicarage, Stone, near Aylesbury, Bucks.
BOURKE, Hon. ALGERNON; London.
BROOKS, Professor E.; Millersville, Pennsylvania.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BURNSIDE, W. S., M.A.; Trinity College, Dublin.
CAMPBELL, Capt. FRED.; Notting Hill, London.
CARR, G. S.; Caius College, Cambridge.
CARNT, E. C.; Royal Hospital School, Greenwich.
CASEY, JOHN, LL.D., F.R.S.; Prof. of Higher Mathematics in the Catholic Univ. of Ireland.
CAVALLIN, C. B. S.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAYLEY, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge; Member of the Institute of France, &c.
CHADWICK, W.; Fellow of C. Ch. Coll., Oxford.
CLARKE, Colonel, C.B., F.R.S.; Director of the Ordnance Survey, Southampton.
CLARKE, MARY A.; Birmingham.
CLIFFORD, W. K., M.A., F.R.S.; late Fellow of Trinity College, Cambridge; Prof. of Applied Mathematics in University College, London.
COCKLE, Hon. Sir J., Knt., M.A., F.R.S.; Chief Justice of Queensland; President of the Queensland Philosophical Society.
COHEN, ARTHUR, M.A., Q.C.; London.
COLLINS, JOHN, B.A.; Marlboro' Street, Dublin.
COLLINS, MATTHEW, LL.D.; Eden Quay, Dublin.
COLQUHOUN, MILLICENT; Inverness.
COLSON, C. G., M.A.; United Coll., St. Andrews.
CONOLLY, E.; Mountnugent, Ireland.
CORNER, S.; Wellington, Somerset.
COTTERILL, J. H., M.A.; Royal School of Naval Architecture, South Kensington.
COTTERILL, THOS., M.A.; London, late Fellow of St. John's College, Cambridge.
CREMONA, LUIGI; Direttore della Scuola degli Ingegneri, S. Pietro in Vincoli, Rome.
CROFTON, M. W., B.A., F.R.S.; Professor of Mathematics and Mechanics in the Royal Military Academy, Woolwich.
DALE, JAMES; Kingsland Place, Aberdeen.
DAVIS, JOHN; Clonakilty, Cork.
DAVIS, REUBEN; Bradford, Illinois, U. States.
DAVIS, R. F., B.A.; Love Lane, London.
DAVIS, WILLIAM BARRETT, B.A. London.
DAWSON, JAMES; Liverpool.
DAY, Rev. H. G., M.A.; Ludlow.
DINNIS, F. H., M.A.; Westbourne Ter., London.
DOBSON, T., B.A.; Head Master of Hexham Grammar School.
DRACH, S. M., Barnsbury Street, London.
DUPAIN, J.C.; Professeur au Lycée d'Angoulême.
DYER, J. M., B.A.; Cheltenham College.
EASTERBY, W., B.A.; Grammar School, St. Asaph.
EASTON, BELLE; Lockport, New York.
EDMUNDSON, GEORGE; Brasenose Coll., Oxford.
ELLIOTT, E.B., M.A.; Fellow of Queen's Coll., Oxon.
ESCOTT, ALBERT, M.A.; Head Master of the Royal Hospital School, Greenwich.
EVANS, Prof. A. B., M.A.; Lockport, New York.
EVERETT, J. D., D.C.L.; Professor of Natural Philosophy in the Queen's University, Belfast.
FICKLIN, JOSEPH; Professor of Mathematics and Astronomy in the University of Missouri.
FORDE, S., M.A.; Oxford.
FORTEY, H., M.A.; Bellary, Madras Presidency.
FRY, Colonel JOHN H.; New York.
FUORTES, E.; University of Naples.
GALTON, FRANCIS, M.A., F.R.G.S.; London.
GARDINER, MARTIN; late Professor of Mathematics in St. John's College, Sydney.
GENESE, R. W., B.A.; Scholar of St. John's College, Cambridge.
GERRANS, H. T., B.A.; Bristol.
GLAISHER, J. W. L., M.A., F.R.S.; Fellow of Trinity College, Cambridge.
GLASHAN, J. C.; Strathroy, Ontario, U. States.
GODFRAY, HUGH, M.A.; Newnham, Cambridge.
GODWARD, WILLIAM; Chelsea.
GRANT, S. C. W.; Royal Naval School, New Cross.
GREAVES, MYRA E.; Rue de la Paix, Boulogne.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITH, W.; Superintendent of Public Schools, New London, Ohio, United States.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
HALL, ASAPH, M.A.; Observatory, Washington.
HAMMOND, J.; King Edward's School, Bath.
HANNA, W.; Literary Institute, Belfast.
HARKEMA, C.; University of St. Petersburgh.
HARLEY, Rev. ROBERT, F.R.S.; Vice-Master of Mill Hill Grammar School.
HARRIS, H. W., B.A.; Trinity College, Dublin.
HART, Dr. DAVID S.; Stonington, Connecticut.
HART, H.; R.M. Academy, Woolwich.
HERMITE, CH.; Membre de l'Institut, Paris.
HILARY, H., M.A.; Dry Hill Park, Tonbridge.
HILL, Rev. E., M.A.; St. John's College, Camb.
HINTON, C. H.; Savile Row, London.
HIRST, Dr. T. A., F.R.S.; Director of Studies in the Royal Naval College, Greenwich.
HOGG, W., M.A.; Royal Naval School, New Cross.
HOPKINS, Rev. G. H., M.A.; Egham, Surrey.
HOPKINSON, J., D.Sc., B.A., Manchester.
HOSKINS, HENRY; Notting Hill Square, London.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Fellow of St. John's College, Cambridge.
INGLEBY, C. M., M.A., LL.D., London.
INGLEBY, O. E.; Jersey.
JAMESON, JOHN; Essex Bridge Works, Dublin.
JENKINS, MORGAN, M.A.; late Scholar of Christ's Coll., Camb., Westbourne Park, London,
JENKINS, J. S., L.C.P.; Malvern.
JOHNSON, J. M., B.A.; Radley College, Abingdon.
JOHNSON, W. W.; Annapolis, Maryland.
JOHNSTONE, E.; Chaplain, Southsea.
JONES, L. W., B.A.; Merton College, Oxford.
KELLAND, PHILIP, M.A.; Professor of Mathematics in the University of Edinburgh.

KING, Q. W.; Royal Hospital School, Greenwich.
KIRKMAN, Rev. T. P., M.A., F.R.S.; Croft Rectory, near Warrington.
KITCHIN, Rev. J. L., M.A.; Head Master of Exeter Grammar School.
KITTUDGE, LIZZIE A.; Boston, United States.
KNISELY, Rev. U. J.; Newcomerstown, Ohio.
LADD, CHRISTINE; Union Springs, New York.
LAVERTY, W. H., M.A.; Public Examiner in the University of Oxford.
LEUDESDORF, C., M.A.; Fellow of Pembroke College, Oxford.
McCAY, W. S., M.A.; Trinity College, Dublin.
McCOLL, HUGH; Rue du Temple, Boulogne.
McCORMICK, E.; Ledbury, Hereford.
McDOWELL, J., M.A.; Pembroke Coll., Camb.
McLEOD, J., M.A.; R.M. Academy, Woolwich.
MACKENZIE, J. L., B.A.; Independent College, Taunton.
MADDEN, W. M.; Trinity Parsonage, Wakefield.
MALET, J. C., M.A.; Trinity College, Dublin.
MANNHEIM, M.; Professeur à l'Ecole Polytechnique, Paris.
MARTIN, ARTEMAS; "Editor of Higher Mathematics" in *Normal Monthly*, Erie.
MARTIN, Rev. H., D.D., M.A.; Examiner in Mathematics in the University of Edinburgh.
MASON, J.; East Castle Colliery, Newcastle.
MATHEWS, F. C., M.A.; London.
MATTESON, Dr. JAMES; De Kalb Centre, Illinois.
MERRIFIELD, C. W., F.R.S.; London.
MERRIFIELD, J., LL.D., F.R.A.S.; Plymouth.
MERRICK, THOS.; Kensington Square, London.
MILLER, W. J. C., B.A.; 55, Netherwood Road, West Kensington Park, London, W.
MINCHIN, G.M., M.A.; Prof. in Cooper's Hill Coll.
MONCK, HENRY STANLEY; Trinity Coll., Dublin.
MOON, ROBERT, M.A.; late Fellow of Queen's College, Cambridge, Cleveland Sq., London.
MORLEY, THOS., L.C.P.; Bromley, Kent.
MOULTON, J. F., M.A.; Fellow of Christ's Coll., Cambridge.
MULCASTER, J.; Woolwich Common.
MURPHY, HUGH; Head Master of the Incorporated Society's School, Dublin.
NARENDRA LAL DEY; Presidency Coll., Calcutta.
NASH, A. M., B.A.; Professor of Nat. Phil. and Astronomy, Presidency College, Calcutta.
NELSON, R. J., M.A.; Naval School, London.
O'CAVANAGH, PATRICK; Dublin.
O'REGAN, JOHN; Limerick.
OGILVIE, G. A.; Leiston, Saxmundham.
OMAKYA CHAKRAVARTI; Presidency College, Calcutta.
OTTER, W. CURTIS, F.R.A.S.; London.
PANTON, A. W., M.A.; Trinity College, Dublin.
PHILLIPS, F. B. W.; Balliol College, Oxford.
PIRIE, A., M.A.; Univ. of St. Andrews.
POLIGNAC, Prince CAMILLE DE; Paris.
PRINCE, J.; Andover, Hants.
PRUDDEN, FRANCES E.; Lockport, New York.
RENSHAW, S. A.; Nottingham.
RHODES, ELLEN; Helensborough.
RICHARDS, J., M.A.; London.
RILEY, R. E., B.A., Bournemouth.
RIPPIN, CHARLES R., M.A.; Woolwich Common.
ROBERTS, R. A., B.A.; Trin. Coll., Dublin.
ROBERTS, SAMUEL, M.A.; Tufnell Pk., London.
ROBERTS, Rev. W., M.A.; Fellow and Senior Tutor, Trinity College, Dublin.
ROBERTS, W., Junior; Trinity College, Dublin.
ROSENTHAL, L. H.; Scholar of Trinity College, Dublin.
RÜCKER, A. W., B.A.; Professor of Mathematics in the Yorkshire College of Science, Leeds.
RUTTER, EDWARD; Sunderland.
SALMON, Rev. G., D.D., F.R.S.; Regius Professor of Divinity in the University of Dublin.
SANDERS, J. B.; Bloomington, Indiana.
SANDERSON, GEORGE; Galashiels.
SANDERSON, Rev. T. J., M.A.; Royston, Cambs.
SARKAR, NILKANTHA, B.A.; Presidency College, Calcutta.
SAVAGE, THOMAS, M.A.; Fellow of Pembroke College, Cambridge.
SCOTT, JOSIAH; Judge of the Ohio Supreme Court, Bucyrus, United States.
SCOTT, R. F., B.A.; St. John's College, Camb.
SEITZ, E. B.; Greenville, Ohio, United States.
SERRET, PAUL; Paris.
SHEPHERD, A. J. P.; Queen's College, Oxford.
SIDES, J. J.; Rue des Vieillards, Boulogne.
SIVERLY, WALTER; Oil City, Pennsylvania.
SLADE, J. H.; Royal Hospital School, Greenwich.
SPOTTISWOODE, WILLIAM, M.A., F.R.S.; Grosvenor Place, London.
STEIN, A.; Venice.
STEPHEN, ST. JOHN, B.A.; Caius Coll., Cambridge.
SYMES, R. W., B.A.; London.
SYLVESTER, J. J., LL.D., F.R.S.; Member of the Institute of France, &c.
TAIT, P. G., M.A.; Professor of Natural Philosophy in the University of Edinburgh.
TANNER, H. W. L., M.A.; Prof. of Mathematics and Physics, R. A. College, Cirencester.
TARLETON, FRANCIS A., M.A.; Fellow of Trinity College, Dublin.
TAYLOR, Rev. C., M.A.; Fellow of St. John's College, Cambridge.
TAYLOR, H. M., M.A.; Fellow and Assistant Tutor of Trinity College, Cambridge.
TAYLOR, J. H., B.A.; Cambridge.
TEBAY, SEPTIMUS, B.A.; Kearsley, Bolton.
THOMSON, F. D., M.A.; late Fellow of St. John's Coll., Camb., Brinkley Rectory, Newmarket.
TIMOTHY, J. E.; Lockport, New York.
TODHUNTER, ISAAC, F.R.S.; Cambridge.
TOMLINSON, H.; Christ Church College, Oxford.
TORELLI, GABRIEL; University of Naples.
TORRY, Rev. A. F., M.A.; St. John's College, Cambridge.
TOWNSEND, Rev. R., M.A., F.R.S.; Professor of Nat. Phil. in the University of Dublin, &c.
TRAILL, ANTHONY, M.A., M.D.; Fellow of Trinity College, Dublin.
TROWBRIDGE, DAVID; Waterburgh, New York.
TUCKER, R., M.A.; Mathematical Master in University College School, London.
TURRELL, I. H.; Cumminsville, Ohio.
VOSE, G. B.; Professor of Mechanics and Civil Engineering, Washington, United States.
WALKER, J. J., M.A.; Vice-Principal of University Hall, Gordon Square, London.
WALMSLEY, J., B.A.; Eccles, Manchester.
WARD, ISABELLA M.; Capecure, Boulogne,
WARREN, R., M.A.; Trinity College, Dublin.
WATSON, STEPHEN; Haydonbridge.
WATSON, Rev. H. W., M.A.; late Fellow of Trinity College, Cambridge.
WERTSCH, Fr.; Weimar.
WHITE, Rev. J., M.A.; Woolwich.
WHITWORTH, Rev. W. A., M.A.; Fellow of St. John's College, Cambridge.
WILKINSON, Rev. M. M. U.; Norwich.
WILLIAMS, S. F; Liverpool College.
WILLIAMSON, B., M.A.; Fellow and Tutor of Trinity College, Dublin.
WILSON, J. M., M.A.; Rugby School.
WILSON, Rev. J., M.A.; Rector of Bannockburn Academy.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WILSON, Rev. R., D.D.; Chelsea.
WOLSTENHOLME, Rev. J., M.A.; Professor of Mathematics in Cooper's Hill College.
WOOLHOUSE, W. S. B., F.R.A.S., &c.; London.
WORDINGHAM, ELIZABETH; Fakenham.
WORTH, J. E.; Borough Offices, Nottingham.
WRIGHT, Dr. S. H., M.A.; Penn Yan, New York.
WRIGHT, W. E.; Royal School, Raphoe.
YOUNG, J. R.; London.

CONTENTS.

Mathematical Papers, &c.

Solved Questions.

CORRIGENDA.

VOL. XXIV.

Page v, line 4 from bottom, *for* $4xa^3$ *read* $4ax^3$.

Page 35, line 21, *for* $4xa^3$ *read* $4ax^3$.

Page 59, line 22, *for* $\frac{dy}{dx} - p(x, y)$ *read* $\frac{dy}{dx} = p(x, y)$.

Page 92, lines 6, 7, 8, *for* z *read* y.

Page 110, line 8, eq. 2, *for* $x^2 - Ay^2 = 1$, *read* $x^2 - Ay^2 = -1$.

Page 111, line 11, *for* $u = x^2 + 1$, *read* $u = 2x^2 + 1$.

VOL. XXV.

Page 96, line 21, *for* $x(\cos\theta \ldots + y)(\ \ )$, *read* $x(\cos\theta - \sin\theta) + y(\ \ )$.

Page 96, line 22, *for* $x(\cos\phi \ldots + z)(\ \ )$, *read* $x(\cos\phi - \sin\phi) + z(\ \ )$.

VOL. XXVI.

Page 53, line 18, *for* September No., *read* page 50.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

5001. (By T. Cotterill, M.A.)—To a system of five points on a conic, show that there is a point upon the conic such that the line through it parallel to the chord joining two of the points passes through the remaining intersection of the conic and circle through the remaining three points.

Solution by Dr. Hirst, F.R.S.

The above theorem is a special case of the following well known one:—

Every cubic which passes through seven fixed points on a given cubic cuts the latter again in two points, which are in line with a fixed eighth point on the given cubic [co-residual to the seven points, see Salmon's *Higher Plane Curves*, 2nd edition, p. 134].

In the special case under consideration the given cubic breaks up into the conic through the five points and the line at infinity; the latter points and the two circular points at infinity being the seven fixed points upon it.

The circle through any three of the five points, and the line through the remaining two, together constitute another cubic, through the seven points whose remaining two intersections with the given one obviously lie on the line drawn through the fourth intersection of the conic and circle, parallel to the line through the two points. This parallel line, therefore, must, by the theorem on cubics above stated, cut the conic again in a fixed point,—the point, in fact, whose existence is predicated in the question under consideration.

It may be observed, by way of corollary, that through this same fixed point the line will pass which is drawn parallel to the asymptote of any circular cubic, passing through the five given points, from its sixth intersection with the conic.

4996. (By Professor CLIFFORD, F.R.S.)—If the series

$$1 + a\frac{1-x}{1-r} + a^2\frac{1-x\,.\,1-rx}{1-r\,.\,1-r^2} + a^3\frac{1-x\,.\,1-rx\,.\,1-r^2x}{1-r\,.\,1-r^2\,.\,1-r^3} + \ldots$$

be called $\phi(a, x)$; then prove that $\phi(1, a)\,.\,\phi(a, x) = \phi(1, ax)$.

Solution by C. LEUDESDORF, M.A.

Writing rx for x, and subtracting, we have

$$\phi(a, rx) - \phi(a, x) = 1 + a\,\frac{1-rx}{1-r} + a^2\frac{(1-x)(1-rx)}{(1-r)(1-r^2)} + \ldots\ldots$$

$$- 1 - a\,\frac{1-x}{1-r} - a^2\frac{(1-x)(1-rx)}{(1-r)(1-r^2)} - \ldots\ldots$$

$$= ax + a^2x\,\frac{1-rx}{1-r} + \ldots\ldots = ax\,\phi(a, rx);$$

therefore $$\phi(a, rx) = \frac{1}{1-ax}\,\phi(a, x).$$

Writing 1 for a, and ax for x, we have similarly

$$\phi(1, arx) = \frac{1}{1-ax}\,\phi(1, ax);\ \text{therefore}\ \frac{\phi(1, ax)}{\phi(a, x)} = \frac{\phi(1, arx)}{\phi(a, rx)};$$

whence we obtain $\phi(1, ax) = C\,\phi(a, x)$,

where C is a constant. Now let $x = 1$; then we have

$$\phi(1, a) = C\,\phi(a, 1) = C;\ \text{therefore}\ \phi(1, ax) = \phi(1, a)\,\phi(a, x).$$

4947. (By Professor TOWNSEND, F.R.S.)—A solid ellipsoid of uniform density being supposed to attract, according to the ordinary law of the inverse square of the distance, an internal particle situated at a given distance from its centre, on one of the four diameters that make equal angles with its three axes of figure; show that the component of the attraction towards the centre is independent of the form as well as of the magnitude of the ellipsoid, and the same as for a sphere of equal density on an internal particle situated at the same distance from its centre.

Solution by the PROPOSER.

For, since for any internal point (x, y, z), employing the ordinary notation, $X = -\mu f\rho\,.\,A\,.\,x$, $Y = -\mu f\rho\,.\,B\,.\,y$, $Z = -\mu f\rho\,.\,C\,.\,z$, where A, B, C are known constants depending on the form of the surface,

and consequently $\quad R = -\mu f\rho\,[Ax^2 + By^2 + Cz^2]\,\frac{1}{r}$,

where r and R are respectively the distance from and attraction towards the centre, therefore, when $x^2 = y^2 = z^2 = \frac{1}{3}r^2$, and when consequently

$$R = -\tfrac{1}{3}\mu f\rho\,[A + B + C]\,r,$$

since always $A + B + C = 4\pi$, then $R = -\frac{4}{3}\pi\mu f\rho \,.\, r$, and therefore, &c.

N.B.—For an *external* point (x_1, y_1, z_1), situated at the distance r_1 on one of the same diameters produced, the value of the component would be

$$R_1 = -\tfrac{4}{3}\pi\mu f\rho \,.\, \frac{abc}{a_1 b_1 c_1} \,.\, r_1,$$

where a, b, c and a_1, b_1, c_1 are respectively the semi-axes of the given ellipsoid and of the confocal to it through the point.

NOTE ON QUESTION 4793 (*Reprint*, Vol. XXIV., page 72).

By Professor WOLSTENHOLME.

The result given in this question is an immediate consequence of the following equality $\quad (xD)^r D^n y = D^n (xD - n)^r y,$

where D denotes $\frac{d}{dx}$, and y is any function of x.

Now $(xD)^r\, Dy$ becomes, on writing $x = \epsilon^\theta$,

$$\left(\frac{d}{d\theta}\right)^r \left(\epsilon^{-\theta}\frac{dy}{d\theta}\right) = \epsilon^{-\theta}\left(\frac{d}{d\theta}-1\right)^r \frac{dy}{d\theta},$$

$$\text{since } f\left(\frac{d}{d\theta}\right)(\epsilon^{m\theta} u) = \epsilon^{m\theta} f\left(\frac{d}{d\theta}+m\right) u$$

$$= \epsilon^{-\theta}\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)^r y = \frac{d}{dx}\left(x\frac{d}{dx}-1\right)^r y = D\,(xD-1)^r y.$$

Assume that for one value of n, for all values of r,

$$(xD)^r D^n y = D^n (xD-n)^r y;$$

then, putting $\frac{dy}{dx}$ for y on both sides, we have

$$(xD)^r D^{n+1} y = D^n (xD-n)^r Dy$$

$$= \epsilon^{-n\theta}\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\cdots\left(\frac{d}{d\theta}-n+1\right)\left(\frac{d}{d\theta}-n\right)^r\left(\epsilon^{-\theta}\frac{dy}{d\theta}\right)$$

$$= \epsilon^{-n\theta}\,.\,\epsilon^{-\theta}\left(\frac{d}{d\theta}-1\right)\left(\frac{d}{d\theta}-2\right)\cdots\left(\frac{d}{d\theta}-n\right)\left(\frac{d}{d\theta}-n-1\right)^r\frac{dy}{d\theta}$$

$$= \epsilon^{-(n+1)\theta}\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\cdots\left(\frac{d}{d\theta}-n\right)\left(\frac{d}{d\theta}-n-1\right)^r y$$

$$= D^{n+1}(xD-n-1)^r y;$$

or, if the theorem be true for any one value of n, it is true for the next greater value, whence &c., as usual.

Now $(x\mathrm{D})^r u = \Sigma_{p=1}^{p=r} \dfrac{\Delta^{p-1} 1^{r-1}}{\lfloor p-1} x^p \dfrac{d^p u}{dx^p}$, whence the result of Quest. 4793;

$$\mathrm{D}^n (x\mathrm{D}-n)^r y = (x\mathrm{D})^r \mathrm{D}^n y = \Sigma_{p=1}^{p=r} \frac{\Delta^{p-1} 1^{r-1}}{\lfloor p-1} x^p \frac{d^{n+p} y}{dx^{n+p}}.$$

$$= \Sigma_{p=1}^{p=r} \frac{\Delta^p 0^r}{\lfloor p} x^p \frac{d^{n+p} y}{dx^{n+p}}.$$

This mode of proof gives the reason for the especially (at first sight) perplexing result, that the coefficients are functions of r only.

[Professor CAYLEY's Solution of the Question is given on p. 72 of Vol. XXIV. of the *Reprint*.]

5004. (By C. LEUDESDORF, M.A.)—A White piece is taken at random and put down on a random square of a clear chessboard, and the same is done with a Black piece; find the chance that White, having the move, will be able to take Black's piece with his.

Solution by S. TEBAY, B.A.

Let n be put for 8, and first consider the pawn. The number of squares that can be attacked from each of the first $n-2$ rows is $2(n-1)$, or $2(n-1)(n-2)$ in all. The chance of drawing a pawn is $\frac{1}{2}$, and the chance of taking $= \dfrac{1}{n^2} \cdot \dfrac{1}{n^2-1} \cdot \frac{1}{2} \cdot 2(n-1)(n-2) = \dfrac{n-2}{n^2(n+1)} = \dfrac{1}{96}$.

If the legitimate range of the pawns be only considered, the chance is

$$\frac{n-3}{n^3-n^2-n} = \frac{1}{88}.$$

The rook commands $2(n-1)$ squares on every square, or $2n^2(n-1)$ in all; and the chance of taking

$$= \frac{1}{n^2} \cdot \frac{1}{n^2-1} \cdot \frac{1}{8} \cdot 2n^2(n-1) = \frac{1}{4(n+1)} = \frac{1}{36}.$$

If the knight be placed on the first row and passed round the board, it will command $16(n-2)$ squares. If on the second row, it will command $8(3n-10)$ squares. Every other position will command 8 squares, or $8(n-4)^2$ in all. Total $= 8(n-1)(n-2)$, and the chance of taking

$$= \frac{1}{n^2} \cdot \frac{1}{n^2-1} \cdot \frac{1}{8} \cdot 8(n-1)(n-2) = \frac{n-2}{n^2(n+1)} = \frac{1}{96}.$$

Let the bishop be placed on the xth square of a diagonal, and moved laterally to the next diagonal, and so round to the first position; it will command $n+2x-3$ squares in each position, and

$$4(n+2x-3)(n-2x+2)-4(n+2x-3)=4(n+2x-3)(n-2x+1)=u_x$$

for the whole circuit. The sum $=\Sigma_1^{\frac{1}{2}n}(u_{x+1})=\frac{2}{3}n(n-1)(2n-1)$; and the chance of taking

$$=\frac{1}{n^2}\cdot\frac{1}{n^2-1}\cdot\frac{1}{8}\cdot\frac{2}{3}n(n-1)(2n-1)=\frac{2n-1}{12n(n+1)}=\frac{5}{288}.$$

If the queen be treated in the same way as the bishop, it will command $3n+2x-5$ squares in each position, and

$$4(3n+2x-5)(n-2x+2)-4(3n+2x-5)=4(3n+2x-5)(n-2x+1)=u_x$$

for the whole circuit. The sum

$$=\Sigma_1^{\frac{1}{2}n}(u_{x+1})=\frac{2}{3}n(n-1)(5n-1);$$

and the chance of taking

$$=\frac{1}{n^2}\cdot\frac{1}{n^2-1}\cdot\frac{1}{16}\cdot\frac{2}{3}n(n-1)(5n-1)=\frac{5n-1}{24n(n+1)}=\frac{13}{576}.$$

If the king be moved round the side-squares of the board, it will command $4(5n-7)$ squares. On any other square it will command 8 squares, or $8(n-2)^2$ in all. The total $=4(n-1)(2n-1)$; and the chance of taking

$$=\frac{1}{n^2}\cdot\frac{1}{n^2-1}\cdot\frac{1}{16}\cdot 4(n-1)(2n-1)=\frac{2n-1}{4n^2(n+1)}=\frac{5}{768}.$$

Therefore the probability required

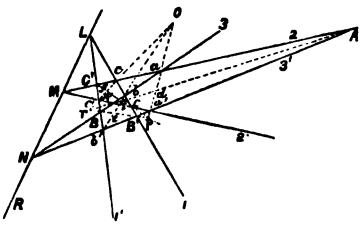
$$=\frac{1}{96}+\frac{1}{36}+\frac{1}{96}+\frac{5}{288}+\frac{13}{576}+\frac{5}{768}=\frac{73}{768}.$$

If we take the restricted value $\frac{1}{88}$ for the pawn, the chance is $\frac{811}{8448}$.

[The values of these several probabilities of check with the pieces agree with those investigated in the Solution of Question 3314, on pp. 50—51 of Vol. XV. of the *Reprint*.]

4855. (By the Rev. F. D. THOMSON, M.A.)—A conic A is described about a hexagon of which the successive sides are the six lines, 1, 2, 3, 1′, 2′, 3′, and three other conics B, C, D are described round the hexagons formed by 123′1′2′3, 12′31′2′3′, 12′3′1′23. Show that a common chord of A and B is in the same straight line with a common chord of C and D, and similarly for the other pairs of conics; and also that nine of the chords of intersection form the sides and the lines joining the vertices of two copolar triangles, one of which is inscribed in the other.

Solution by the Proposer.



The equations to the four conics may be written

$$12'3 = \kappa\, 1'23' \;\ldots\ldots\; (A),$$
$$12'3' = \lambda\, 1'23 \;\ldots\ldots\; (B),$$
$$123 = \mu\, 1'2'3' \;\ldots\ldots\; (C),$$
$$123' = \nu\, 1'2'3 \;\ldots\ldots\; (D),$$

the constants $\kappa, \lambda, \mu, \nu$ being determined so that the highest terms may vanish. And, by Pascal's theorem, $11', 22', 33'$ meet in points on a line R. Hence, if we take $x=0,\ y=0,\ z=0$ for the equations of 1, 2, 3, we may take $ax+R=0$, $by+R=0$, $cz+R=0$ for $1', 2', 3'$. The equation (A) will be of the form

$$acx\,(by+R)\,z = b\,(ax+R)\,y\,(cz+R),$$

and similarly for the others.

Hence we get for common chords of

$$\left.\begin{matrix}A\\ \text{and}\\ B\end{matrix}\right\}\begin{matrix}2cz+R=0,\\ \\ ax-by=0,\end{matrix}\qquad \left.\begin{matrix}C\\ \text{and}\\ D\end{matrix}\right\}\begin{matrix}2cz+R=0,\\ \\ ax+by+R=0,\end{matrix}$$

and similarly for the other pair of conics; whence the results stated in the Question immediately follow.

If the angular points of the four hexagons be $BcaB'c'a'$, $AbcA'b'c'$, $BCA'B'C'A$, $CabC'a'b'$, then aa', bb', cc' meet in o; BB', CC', aa' in p; CC', AA', bb' in q; AA', BB', cc' in r. And if rq meet op in d, oq meet pr in e, pq meet or in f, the nine chords spoken of are the sides and connectors of the two copolar triangles pqr, def.

4934. (By Christine Ladd.)—Find the locus of a point when the middle points of the chordal segment of its polar with respect to any circle of a system having a common radical axis is on a fixed circle.

Solution by the Proposer, *the* Rev. H. G. Day, M.A., *and others.*

If (x', y') be the coordinates of the point, the equation of its polar with respect to any circle of the system $x^2+y^2-2hx\pm\delta^2=0$ will be

$$xx'+yy'-h\,(x+x')\pm\delta^2=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

The middle point of the chordal segment will be the intersection of the polar by the line through (x', y') and the centre of the circle, or

$$y-y'=\frac{-y'}{h-x'}\,(x-x') \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

Eliminating h between (1) and (2), we have

$$x^2+y^2-\frac{x'^2\pm\delta^2}{y'}\pm\delta^2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

(3) is the equation of a circle of the system $x^2+y^2-2ky\pm\delta^2=0$, whatever be the position of (x', y'). If the circle is fixed, we have

$$\frac{x'^2\pm\delta^2}{y'}=2p, \text{ or } x'^2=2py'\pm\delta^2;$$

therefore the required locus is a parabola.

4942. (By S. WATSON.)—A semicircle is divided into two portions by an ordinate, and circles are inscribed in those portions: find the average distance of their centres.

Solution by the Rev. H. G. DAY, M.A.

If the abscissa be represented by $a\cos\theta$, then the two radii are

$$2a(\cos\tfrac{1}{2}\theta-\cos^2\tfrac{1}{2}\theta) \text{ and } 2a(\sin\tfrac{1}{2}\theta-\sin^2\tfrac{1}{2}\theta);$$

and if k be the distance between the centres, we have

$$k^2=2(\rho_1^2+\rho_2^2)=8a^2(1+\cos^4\tfrac{1}{2}\theta+\sin^4\tfrac{1}{2}\theta-2\cos^3\tfrac{1}{2}\theta-2\sin^3\tfrac{1}{2}\theta)$$

$$=4a^2\left\{6+3\sin\theta-4(1+\sin\theta)\right\}.$$

The required average is $\displaystyle\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}k\sin\theta\,d\theta.$

5011. (By A. W. CAVE.)—If the circle of curvature at any point P of a parabola meet the parabola again in Q, and (x, y) be the coordinates of P, and ρ the radius of curvature at P, prove that $PQ^3=128xy\rho$.

Solution by R. TUCKER, M.A.; *the* PROPOSER; *and others.*

Let the coordinates of P be $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$; then, since the common chord and the tangent at P are equally inclined to the axis, the equation to PQ is

$$y+mx=\frac{3a}{m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

hence, for the point of section $\left(\frac{a}{m'^2},\ \frac{2a}{m'}\right)$, we have, by (1),

$$3m'+m=0,\ \text{that is,}\ m'=-\tfrac{1}{3}m;$$

therefore $$PQ=\left(\frac{8a}{m^2}\right)^2+\left(\frac{8a}{m}\right)^2=\frac{64a^2}{m^4}(m^2+1);$$

hence $$PQ^3=\frac{512a^3(m^2+1)^{\frac{3}{2}}}{m^6}.$$

Again, since $\rho=2a\sec^3\theta$ is the radial of a parabola (where $\cot\theta=m$), therefore $\rho=\dfrac{2a(1+m^2)^{\frac{3}{2}}}{m^3}$, and $xy=\dfrac{2a^2}{m^3}$; therefore $PQ^3=128\rho xy$.

4890. (By A. B. Evans, M.A.)—Integrate the simultaneous equations

$$u_x-v_x=u_{x-1}+v_{x-1},\quad u_x+v_x=9u_{x-}+11v_{x-1}-4(-1)^x.$$

Solution by R. Tucker, M.A.

Putting the equations in the form

$$u_{x+1}-v_{x+1}=u_x+v_x,\quad u_{x+1}+v_{x+1}=9u_x+11v_x+4(-1)^x,$$

they may be written

$$(D-1)u_x-(D+1)v_x=0,\quad (D-9)u_x+(D-11)v_x=4(-1)^x;$$

therefore $$(D^2-10D+1)u_x=2(D+1)(-1)^x=0;\ \text{hence}$$

$$u_x=c_1(5+2\sqrt{6})^x+c_2(5-2\sqrt{6})^x=c_1(\sqrt{3}+\sqrt{2})^{2x}+c_2(\sqrt{3}-\sqrt{2})^{2x},$$

and $$(D+1)v_x=c_1(4+2\sqrt{6})(5+2\sqrt{6})^x-c_2(2\sqrt{6}-4)(5-2\sqrt{6})^x,$$

$$v_x=\frac{c_1(4+2\sqrt{6})}{6+2\sqrt{6}}(5+2\sqrt{6})^x-\frac{c_2(2\sqrt{6}-4)}{6-2\sqrt{6}}(5-2\sqrt{6})^x+(-1)^x(c_1'+c_2')$$

$$=\frac{c_1(2+\sqrt{6})}{3+\sqrt{6}}(\sqrt{3}+\sqrt{2})^{2x}-\frac{c_2(\sqrt{6}-2)}{3-\sqrt{6}}(\sqrt{3}-\sqrt{2})^{2x}+C(-1)^x.$$

By substitution in (2), the constant C is found to be $-\frac{1}{3}$.

4929. (By the Editor.)—Given three concentric circles, show (1) how to construct geometrically two equilateral triangles which shall have a common vertex on the innermost circumference, and the other vertices lying one on each of the other two; (2) that if a, b, c be the radii of

the circles, the values of the side (x) of the triangles will be given by

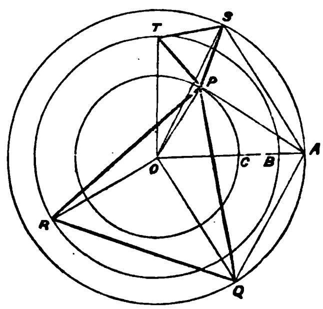
$$\cos^{-1}\left(\frac{a^2+x^2-b^2}{2ax}\right)+\cos^{-1}\left(\frac{a^2+x^2-c^2}{2ax}\right)=\tfrac{1}{3}\pi;$$

(3) that the values of $2x^2$ from this equation are

$$(a^2+b^2+c^2)\pm\left\{3(a+b+c)(b+c-a)(c+a-b)(a+b-c)\right\}^{\frac{1}{2}};$$

and (4), from these values, and also from the geometrical construction, that if $a=b+c$ the two triangles are equal in magnitude, but if $a>b+c$ the construction is impossible.

I. *Solution by the* Proposer.



1. Let O be the common centre of the three circles; P a point on the third or innermost circle; and A a point on the first circle such that PA is equal to OB, the radius of the second circle. From A as centre, with AO as radius, draw a circle cutting the first circumference in Q and S; then PQ, PS will be the sides of the two equilateral triangles required.

From Q and S as centres, with QP and SP respectively as radii, draw circles, and let R and T be the points, remote from A, in which they cut the second circumference; then, drawing the other lines as in the figure, we have the triangle OQR equal-sided with AQP, and therefore the angle OQR is equal to AQP, and the angle PQR equal to AQO; consequently the triangle PQR—and, in like manner, PST—is equilateral.

If AP ($=$ OB) $=$ AC, or OA $=$ OB $+$ OC, the two triangles PQR, PST will be identically equal; and if AP $<$ AC, or OA $>$ OB $+$ OC, no such equilateral triangles can evidently be drawn.

2. We have $\cos \mathrm{OQR}=\dfrac{a^2+x^2-b^2}{2ax}$, and $\cos \mathrm{OQP}=\dfrac{a^2+x^2-c^2}{2ax}$;

whence the equation for x follows at once.

3. From this equation we obtain, successively,

$$\cos^{-1}\left(\frac{a^2+x^2-c^2}{2ax}\right)=\cos^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{a^2+x^2-b^2}{2ax}\right),$$

$$\frac{a^2+x^2-c^2}{2ax}=\frac{a^2+x^2-b^2}{4ax}+\left\{\frac{3}{4}-\frac{3}{4}\left(\frac{a^2+x^2-b^2}{2ax}\right)^2\right\}^{\frac{1}{2}},$$

$$(a^2+x^2-b^2)+2(b^2-c^2)=\left\{12a^2x^2-3(a^2+x^2-b^2)^2\right\}^{\frac{1}{2}},$$

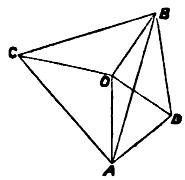
$$(a^2+x^2-b^2)^2+(b^2-c^2)(a^2+x^2-b^2)+(b^2-c^2)^2=3a^2x^2,$$

$$x^4-(a^2+b^2+c^2)x^2=(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4),$$

$$
\begin{aligned}
2x^2 &= (a^2+b^2+c^2) \pm \left\{6\,(b^2c^2+c^2a^2+a^2b^2) - 3\,(a^4+b^4+c^4)\right\}^{\frac{1}{2}} \\
&= (a^2+b^2+c^2) \pm \left\{3\,(a^2+b^2+c^2)^2 - 6\,(a^4+b^4+c^4)\right\}^{\frac{1}{2}} \\
&= 4\,(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}} \cos\left(\tfrac{1}{3}\pi \pm \phi\right), \text{ where } \cos\phi = \frac{a^2+b^2+c^2}{2\,(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}}, \\
&= (a^2+b^2+c^2) \pm \left\{3\,(a+b+c)\,(b+c-a)\,(c+a-b)\,(a+b-c)\right\}^{\frac{1}{2}}.
\end{aligned}
$$

4. From the last form for $2x^2$, as well as from what is given at the end of Art. 1, the results stated in the fourth part of the Question are immediately seen to be true.

II. *Solution by* R. F. Davis, B.A.; A. B. Evans, M.A.; *and others.*



Let O be the common centre and A the given point on the innermost circumference. Upon OA draw the equilateral triangle OAD, and take a point B upon the circumference of the second circle such that BD = c. Construct upon AB the equilateral triangle ABC, observing that its angular point C will lie on the circumference of the third circle; for from the equality of the triangles ACO, ABD, OC = BD = c.

This is one of the required triangles. If B′ be the image of B with respect to OD, the second is the equilateral triangle drawn on AB′.

Since a triangle OBD is employed in the above construction whose sides are a, b, c, it is absolutely essential that a be not greater than $b+c$. When $a = b+c$, both B and B′ lie in OD, and the required triangles coincide.

The first equation for x is sufficiently obvious. The second is to be found from the triangle AOB by substituting the known values of cos BOD and sin BOD in the relation $x^2 = a^2+b^2-2ab\cos(\frac{1}{3}\pi \pm \text{BOD})$.

III. *Solution by* S. A. Renshaw; E. Rutter; *and others.*

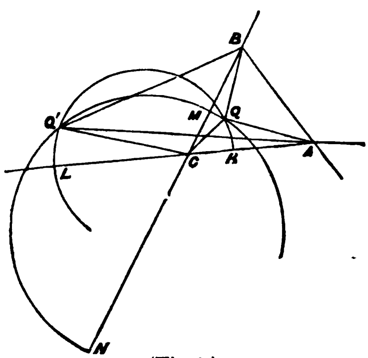
1. Let O (Fig 2) be the common centre of the three circles Oa, Ob, Oc. Take any equilateral triangle CAB, and divide the side AC internally and externally in the points K and L (Fig 1), so that

AK : KC = AL : LC
= Oa : Oc.

Similarly, divide the side BC in M and N so that

BM : MC = BN : NC
= Ob : Oc;

and on KL, MN as diameters draw circles intersecting in Q and Q′.



(Fig 1.)

On the radius OC (Fig 2) take

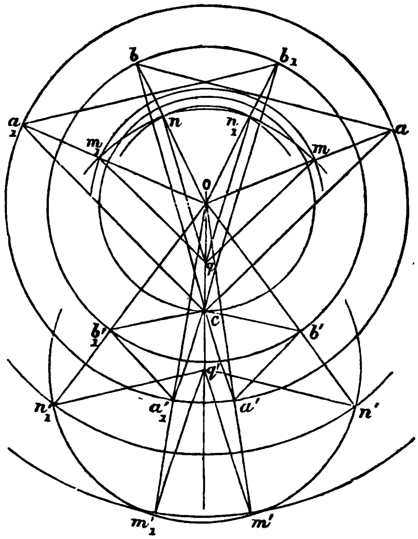
$$Oq = CQ,$$

and $Oq' = CQ.$

With centre O and radii QA, QB, Q′A, Q′B draw circles; with centres q and q' and radius CA draw circles cutting the former in m, m_1, n, n_1, m', m'_1, n', n'_1; join qm, qm_1, qn, qn_1, $q'm'$, $q'm'_1$, $q'n'$, $q'n'_1$; draw ca, ca_1, cb, cb_1, ca', ca'_1, cb', cb'_1, respectively parallel to them, and meeting the given circles as in the figure; then, joining ab, a_1b_1, $a'b'$, $a'_1b'_1$, each of the triangles abc, a_1b_1c, $a'b'c$, $a'_1b'_1c$ satisfies the question.

For, since by construction

$$CQ : BQ : AQ = c : b : a = CQ' : BQ' : AQ',$$



(Fig 2.)

it is evident that the triangles Oca, Ocb, Oba, Oca', Ocb', $Ob'a'$ are similar to CQA, CQB, BQA, CQ′A, CQ′B, BQ′A respectively; whence it follows that $ca = ab = bc$ and $ca' = a'b' = b'c$; therefore the triangles abc, $a'b'c$, &c. are equilateral, and are drawn as required.

2. From the triangles bOc, aOc we have $b^2 = x^2 + c^2 - 2cx \cos bcO$, and $a^2 = x^2 + c^2 - 2cx \cos acO$; whence, since $bca = \frac{1}{3}\pi$, we have

$$\cos^{-1}\left(\frac{x^2+c^2-b^2}{2cx}\right) + \cos^{-1}\left(\frac{x^2+c^2-a^2}{2cx}\right) = \cos^{-1}\left(\tfrac{1}{2}\right).$$

3. From this equation we obtain, successively,

$$\left(\frac{x^2+c^2-b^2}{2cx}\right)\left(\frac{x^2+c^2-a^2}{2cx}\right) - \left\{1-\left(\frac{x^2+c^2-b^2}{2cx}\right)^2\right\}^{\frac{1}{2}}\left\{1-\left(\frac{x^2+c^2-a^2}{2cx}\right)^2\right\}^{\frac{1}{2}} = \tfrac{1}{2},$$

$$-\frac{(x^2+c^2-b^2)(x^2+c^2-a^2)}{4c^2x^2} + \tfrac{1}{4} = 1 - \frac{(x^2+c^2-b^2)^2}{4c^2x^2} - \frac{(x^2+c^2-a^2)^2}{4c^2x^2},$$

$$-\frac{(x^2+c^2)^2-(b^2+a^2)(x^2+c^2)+a^2b^2}{4c^2x^2} = \tfrac{3}{4} - \frac{2(x^2+c^2)^2-2(b^2+a^2)(a^2+c^2)+b^4+a^4}{4c^2x^2},$$

$$(x^2+c^2)^2-(b^2+a^2)(x^2+c^2)+b^4+a^4-a^2b^2 = 3c^2x^2,$$

$$x^4+c^4+2c^2x^2-(b^2+a^2)\,x^2-3c^2x^2-b^2c^2-c^2a^2+b^4+a^4-a^2b^2=0,$$

$$x^4-(a^2+b^2+c^2)\,x^2=(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4);$$

whence the result follows as in the first solution.

4. From these values of $2x^2$ it is evident that when $a=b+c$ the quantity under the radical vanishes, and that then the values of x are each equal to $\frac{1}{2}(a^2+b^2+c^2)^{\frac{1}{2}}$; also when $a>b+c$ the quantity under the radical is negative, and the question becomes therefore impossible.

The same appears from the geometrical construction; for the question admits of two unequal or equal triangles or becomes impossible, according as the circles on LK and MN cut one another in two points, touch one another, or fall one within the other.

5009. (By A. MARTIN.)—Prove (1) that all powers of 12890625 terminate with 12890625; and (2) that all powers of numbers terminating with 12890625 terminate with 12890625.

I. *Solution by* E. B. ELLIOTT, M.A.

1. We have $12890625=5^8.33.$

Thus the difference between the number and its square is

$$\begin{aligned}5^8.33.(5^8.33-1)&=5^8.33.\left\{(1+2^2)^8(1+2^5)-1\right\}\\&=5^8.33.\left\{(1+8.2^2+28.2^4+56.2^6\ldots)(1+2^5)-1\right\}\\&=5^8.33.\left\{(1+2^5+7.2^6+\text{a multiple of }2^8)\right.\\&\qquad\qquad\left.\times(1+2^5)-1\right\}\\&=5^8.33.(2.2^5+7.2^6+\text{a multiple of }2^8)\\&=5^8.33.(\text{a multiple of }2^8)=\text{a multiple of }10^8.\end{aligned}$$

Thus the square of 12890625 terminates with the same 8 digits.

2. Hence, and from the fact that the last p digits of $n\,(M.10^p+n)$ are the same as those of n^2, it follows that the third and all higher powers of 12890625 end with the same digits.

Since also $(M.10^p+n)(M'.10^q+n)$, p being less than q, has the same last p digits as n^2 has, the product of any two numbers each ending in 12890625 also ends in 12890625. This involves the second part of the question.

II. *Solution by* Professor TANNER, M.A.

1. $$12890625=m\,(\text{say})=33\times5^8,$$
$$12890624=m-1=50354\times2^8;$$

therefore $m^r=m+(m^r-m)=m+km\,(m-1)=m+k'.10^8;$

so that m^r terminates in the same eight digits as m.

2. Any numbers terminating in 12890625 may be written

$$n_1 \,.\, 10^8 + m,\ n_2 \,.\, 10^8 + m,\ \ldots\ldots,\ n_r \,.\, 10^8 + m,$$

therefore their product $\equiv \mathrm{N} + 10^8 + m^r$,

and terminates in the same eight digits as m, by (1). Hence the continued product of any numbers terminating in 12890625 also terminates in 12890625.

III. *Solution by* R. TUCKER, M.A.

$$(12890625)^2 = 166168212890625 = 10^8\,\mathrm{Q} + 12890625,$$

that is $\mathrm{P}^2 = 10^8 \,.\, \mathrm{Q} + \mathrm{P}$;

therefore $\mathrm{P}^3 = 10^8\,\mathrm{Q} \,.\, \mathrm{P} + \mathrm{P}^2 = 10^8\,\mathrm{Q}' + \mathrm{P}.$

The first part (for powers of P) is readily seen to be true. The second also follows, substituting R for Q, therefore &c.

4756. (By Prof. TOWNSEND, F.R.S.)—Deduce the following cases of free from the corresponding cases of brachystochronous motion, viz.:—

(*a*.) Show that a cycloid may be described freely under the action of a constant force which envelopes the conterminous cycloid of twice the number of cusps on the same linear base; the velocity of description vanishing at every cusp common to both.

(*b*.) Show that a cardioid may be described freely under the action of a force which envelopes the conterminous epicycloid of twice the number of cusps on the same circular base; the force varying directly as the square of the tangential distance from the envelope, and the velocity of description vanishing at the cusp common to both.

(*c*.) Show that a circle may be described freely under the action of a force which envelopes a bicuspidal epicycloid inscribed to and concentric with it; the force varying directly as the tangential distance from the envelope, and the velocity of description vanishing at each vertex of the epicycloid.

(*d*.) Show that a circle may also be described freely under the action of a force which envelopes a cardioid inscribed to and concentric with it; the force varying directly as the cube of the tangential distance from the envelope, and the velocity of description vanishing at the vertex of the cardioid.

Solution by the PROPOSER.

It is known (see *Reprint*, Vol. XXI., page 56) that, if any curve be a free path for some force, it will be a brachystochronous path, under the same velocity of description, for the reflection of the force with respect to the current tangent to the curve, and conversely. It is known also (see *Quarterly Journal of Mathematics*, Vol. XIV., pages 1 *et seq.*) that the cycloid is brachystochronous, under a velocity of description vanishing at its

base, for a constant force acting perpendicularly from its base; the cardioid, under a velocity of description vanishing at its cusp, for a force acting radially from and varying directly as the square of the distance from the cusp; the circle, under a velocity of description vanishing at any diameter, for a force acting perpendicularly from and varying directly as the distance from the diameter; and the circle again, under a velocity of description vanishing at any point, for a force acting radially from and varying directly as the cube of the distance from the point. And therefore, &c. as regards (*a*), (*b*), (*c*), (*d*); the several envelopes of the several forces for brachystochronism, in the four cases, when reflected for free motion, being the well known *caustics by reflexion* of the four curves for rays of light incident in the directions of the unreflected forces in the several cases; and the tangential reflected distances from the same envelopes varying as the original unreflected distances in all alike.

4987. (By M. Jenkins, M.A.)—If $a_2, a_3 \dots a_3, a_2$ be any reversible series of numbers, show how to determine a_0, a_1 so that $a_0, \dot{a}_1, a_2 \dots a_1, 2\dot{a}_0$ may be the series of quotients of a quadratic surd of the form $N^{\frac{1}{2}}$, the dots marking the recurring period of quotients.

Solution by the Proposer.

Write down the numbers $a_2, a_3 \dots a_2, a_1$; add them up with this modification, that whenever we come to an even sum we strike out the next number, whether even or odd; and, whenever we come to an odd sum, we strike out every immediately succeeding even number. Give a_1 any integral value so as to make the final sum even; it will then be found that a_0 can be found from the equation $2a_0 = mp \pm p'q'$,

$$
\begin{array}{llll}
\text{where } p & \text{denotes the} & \text{cumulant} & (a_1, a_2 \dots a_2, a_1), \\
p' & \quad ,, & \quad ,, & (a_1 \dots a_2), \\
q = p' & \quad ,, & \quad ,, & (a_2 \dots a_1), \\
q' & \quad ,, & \quad ,, & (a_2 \dots a_2);
\end{array}
$$

the upper or lower sign being taken according as the number of quotients in $\dfrac{p}{q}$ is even or odd, and m any integer which will make $mp \pm p'q'$ positive.

Example.—Determine a_0, a_1, so that

$$a_0, \dot{a}_1, 2, 2, 2, 1, 1, 1, 2, [1], 2, 1, 1, 1, 2, 2, 2, a_1, 2\dot{a}_0$$

may be the quotients of the expansion of a surd $N^{\frac{1}{2}}$,

$$2, \cancel{2}, 2, \cancel{1}, 1, 1, \cancel{2}, 1, \cancel{2}, 1, \cancel{1}, 1, \cancel{2}, \cancel{2}, \cancel{2}, a_1.$$

The final sum is $9 + a_1$, which can only be even if a_1 is odd. Let $a_1 = 3$; then a suitable value of a_0 will be found to be 148144, and 148144, 3, 2, 2, 2, &c. ..... 2, 2, 2, 3, 296288 are the quotients of $(148144^2 + 86807)^{\frac{1}{2}}$.

The quotients beginning with 2 were chosen at random, and the quotients of the quadratic surd have been found by the ordinary process to agree with those first chosen.

Proof.—Let $x = \dfrac{(a_0, \dot{a}_1, a_2 \dots a_1, 2\dot{a}_0)}{(\dot{a}_1 \dots a_1, 2\dot{a}_0)}$; then $x^2 = a_0^2 + \dfrac{(2a_0, a_1 \dots a_2)}{(a_1 \dots a_1)}$; then, if $x = \mathrm{N}^{\frac{1}{2}}$, where N is an integer, not a perfect square, $\dfrac{(2a_0, a_1 \dots a_2)}{(a_1, a_2 \dots a_1)}$ will be an integer $< 2a_0 + 1$.

Conversely, if integers a_0, a_1, &c. can be found making $\dfrac{(2a_0, a_1 \dots a_2)}{(a_1 \dots a_1)}$ an integer, say $= t$, then t will be $< 2a_0 + 1$, being

$$= \frac{(2a_0, a_1 \dots a_2)}{(a_1 \dots a_2)} \cdot \frac{(a_1 \dots a_2)}{(a_1 \dots a_1)};$$

that is
$$= \left(2a_0 + \frac{1}{a_1 +}\ \text{\&c.}\right)\left(\frac{1}{a_1 +}\ \frac{1}{a_2 + \dots}\right);$$

also $x = (a_0^2 + t)^{\frac{1}{2}}$, and the quotients in the expansion of this surd can be written down at once.

From the condition stated we have

$$2a_0 \,.\, q + q' = pt, \quad \text{or} \quad p \,.\, t - q \,.\, 2a_0 = q';$$

also
$$pq' - q \,.\, p' = \pm 1;$$

therefore
$$\left.\begin{array}{l} 2a_0 = mp \pm p'q' \\ t = mq \pm q'^2 \end{array}\right\} \begin{array}{l}\text{according as the number of} \\ \text{quotients in } p \text{ is even or odd.}\end{array}$$

If $p'q'$, that is qq', be odd, then $pq' = qp' \pm 1$ must be even; and therefore p must be even, and therefore $mp \pm p'q'$ is odd, and a value cannot be found for $2a_0$. But if qq' be even, any value of m may be taken which will make $mp \pm p'q'$ even and positive. The necessary and sufficient condition therefore for the solution of the problem is that qq' should be even.

The reason of the test given for the evenness of qq' is as follows:—From the laws of continued fractions no two consecutive denominators of the convergents can be even; also, if we use d to denote a consecutive pair of odd denominators, o an odd-even pair, e an even-odd pair; then o is followed by e whatever be the value of the connecting quotient; d is followed by d or o according as the quotient is even or odd, and e is followed by o or d according as the quotient is even or odd; also the first denominator is odd, being $= 1$; whence follows the test given.

4980. (By the Rev. J. Blissard, B.A.)—Prove that

$$\left(\frac{n}{1} - \frac{n(n-1)}{1.2} \,.\, 2 + \frac{n(n-1)(n-2)}{1.2.3} \,.\, 2\right)$$
$$- \left(\frac{n(n-1) \dots (n-4)}{1.2.3.4.5} \,.\, 2^2 - \frac{n \dots (n-5)}{1.2 \dots 6} \,.\, 2^3 + \frac{n \dots (n-6)}{1.2 \dots 7} \,.\, 2^3\right)$$
$$+ \left(\frac{n \dots (n-8)}{1.2 \dots 9} \,.\, 2^4 - \dots\dots\right) + \dots\dots = 0 \ \text{if } n \text{ is even,}$$

and $= \pm 1$ as n is of the form $4m \pm 1$.

Solution by J. HAMMOND, B.A.

$$\text{If}\quad u=\left(n-\frac{[n]^2}{|\underline{2}}\,.\,2+\frac{[n]^3}{|\underline{3}}\,.\,2\right)-\left(\frac{[n]^5}{|\underline{5}}\,.\,2^2-\frac{[n]^6}{|\underline{6}}\,.\,2^3+\frac{[n]^7}{|\underline{7}}\,.\,2^3\right)+\&\text{c.},$$

$$\Delta^4 u=-2^2 u,\quad \text{therefore}\quad \Delta=\surd 2\,(-1)^{\frac{1}{4}}=\pm(1\pm\sqrt{-1}),$$

and $\quad E=1+\Delta=2\pm\sqrt{-1}$ or $\pm\sqrt{-1}$; hence we have

$$u=A\,5^{\frac{1}{2}n}\cos(n\tan^{-1}\tfrac{1}{2})+B\,5^{\frac{1}{2}n}\sin(n\tan^{-1}\tfrac{1}{2})+C\cos\tfrac{1}{2}n\pi+D\sin\tfrac{1}{2}n\pi.$$

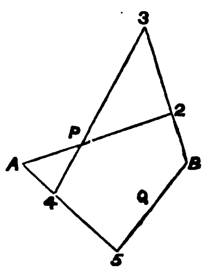
Putting $n=0, 1, 2, 3$ to determine A, B, C, D, we have

$$A+C=0,\quad 2A+B+D=1,\quad 3A+4B-C=0,\quad 2A+11B-D=-1;$$

hence $A=0$, $B=0$, $C=0$, $D=1$; and $u=\sin\frac{1}{2}\pi=0$, when n is even, $=\pm 1$ as n is of the form $5m\pm 1$.

4869. (By J. J. SYLVESTER, F.R.S.)—To draw a tangent to any cubic curve. Let A, B be any two given points of inflexion, and P the given point. Let AP meet the curve in 2, B2 in 3, P3 in 4, A4 in 5, B5 in Q; then the line PQ touches the curve at P; or, as we may express the theorem symbolically, PBAPBAP = P, or, more symbolically still, $(BAP)^2=1$.

In the illustrative figure, A, B being points of inflexion, and P, 2, 3 4, 5, Q six other points on the curve, the line PQ touches the curve at P. Required to prove the above construction.



Solution by J. L. MCKENZIE.

A proof of this construction may be derived from the theorem:—"If any cubic pass through seven given points on a given cubic, then the line joining the two remaining intersections of the two cubics will cut the given cubic again in another fixed point—the co-residual of the seven given points."—(See Salmon's "Higher Plane Curves," 2nd Ed.)

If P, P, A, A, 4, 4, 5 be taken as the seven given points, the lines AP, 4P, and A45, will form one degenerate system of the third order through the seven given points; therefore 32 passes through B, the co-residual of the given points. But the lines PP, A45, and A4 (taken a second time) will form another degenerate cubic, cutting the given cubic again in 5 and Q; therefore 5Q passes through B. Whence the construction:—Draw *any* line cutting the cubic in A, 4, 5; join AP and 4P, cutting the cubic in 3 and 2; draw 32 cutting the cubic in B; join 5B cutting the cubic in Q. Then PQ is the tangent at P. From this proof it is seen that A and B are not necessarily points of inflexion.

5032. (By Rev. A. F. TORRY, M.A.)—Before a class list is published the probability that the classes will be in order of merit is p. When it is published there are a names in the first class, b in the second, and c in the third, the names in each class being in alphabetical order: find the chance that this is also the order of merit.

Solution by the Rev. H. G. DAY, M.A.

The chance of the arrangement being in order of merit is to the chance of it being alphabetical as $p \cdot \dfrac{1}{\underline{|a}} \cdot \dfrac{1}{\underline{|b}} \cdot \dfrac{1}{\underline{|c}} : 1-p.$

Hence the chance required is $p \div \{p+(1-p)\,\underline{|a}\;\underline{|b}\;\underline{|c}\}$.

This is solved, excluding the possibility that two classes should be arranged in order of merit and one alphabetical.

4974. (By Prof. WOLSTENHOLME, M.A.)—An ellipse of constant excentricity $\dfrac{2\lambda}{1+\lambda^2}$ passes through the focus of the parabola $y^2 = 4ax$, and its own real foci lie on the parabola; prove that its major axis is a tangent to one of the two parabolas (confocal with the given one)

$$y^2 = 4a(1-\lambda^2)(x-a\lambda^2);$$

and its minor axis is always normal to one of the two

$$y^2 = 4a(1+\lambda^2)(x+a\lambda^2),$$

where λ has two values.

Solution by the PROPOSER; L. W. JONES, B.A.; *and others.*

Suppose the equation of the major axis of the ellipse to be $y = mx+c$, and let $(x_1 y_1)$, $(x_2 y_2)$ be the points where this meets the parabola $y^2=4ax$, or the foci of the ellipse.

Then y_1, y_2 are the roots of the equation

$$y^2 = \frac{4a}{m}(y-c); \quad \text{or} \quad (y_1-y_2)^2 = \frac{16a(a-cm)}{m^2},$$

and

$$(x_1-x_2)^2+(y_1-y_2)^2 = \frac{16a(a-cm)(1+m^2)}{m^4}.$$

Again, the major axis

$$= 2a+x_1+x_2 = 2a+\frac{y_1+y_2-2c}{m} = 2a+\frac{4a}{m^2}-\frac{2c}{m}.$$

These will be simpler in form if we assume $c = \dfrac{a(1-p^2)}{m}$, when we shall

have the distance between the foci of the ellipse $\frac{4ap}{m^2}(1+m^2)^{\frac{1}{2}}$, and its major axis $= \frac{2a(1+m^2+p^2)}{m^2}$. Hence

$$\frac{2p(1+m^2)^{\frac{1}{2}}}{p^2+1+m^2} = \text{eccentricity} = \frac{2\lambda}{1+\lambda^2}, \text{ or } \frac{p}{(1+m^2)^{\frac{1}{2}}} = \lambda,$$

where λ has either root of the equation $\frac{2\lambda}{1+\lambda^2} = e$.

The equation of the major axis is then

$$y = mx + \frac{a}{m} - a(1+m^2)\frac{\lambda^2}{m} = m(x-a\lambda^2) + \frac{a(1-\lambda^2)}{m},$$

or is always a tangent to the parabola $y^2 = 4a(1-\lambda^2)(x-a\lambda^2)$.

The centre of the ellipse is

$$\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}, \text{ or } \frac{y_1+y_2-2c}{2m},\ \frac{y_1+y_2}{2},$$

or
$$\frac{2a}{m^2} - \frac{a}{m^2}\left\{1-\lambda^2(1+m^2)\right\},\ \frac{2a}{m};$$

and the equation of the minor axis is

$$y - \frac{2a}{m} = -\frac{1}{m}\left\{x - \frac{a}{m^2} - \frac{a\lambda^2(1+m^2)}{m^2}\right\},$$

or
$$y = -\frac{1}{m}\left\{x - a - a(1+\lambda^2)\left(1+\frac{1}{m^2}\right)\right\},$$

which is normal to the parabola $y^2 = 4a(1+\lambda^2)(x+a\lambda^2)$ at the point

$$x + a\lambda^2 = \frac{a(1+\lambda^2)}{m^2},\quad y = \frac{2a(1+\lambda^2)}{m}.$$

Of course the straight line joining this to the point of contact of the major axis with its envelope passes through the focus.

The equation of the *ellipse* is

$$\frac{\left\{my + x - a - a(1+\lambda^2)\left(1+\frac{1}{m^2}\right)\right\}^2}{(1+\lambda^2)^2} + \frac{\left\{y - m\left(x - a\lambda^2 + \frac{a}{m^2}(1-\lambda^2)\right)\right\}^2}{(1-\lambda^2)^2} = \frac{a^2(1+m^2)^3}{m^4};$$

and if we move the origin to the focus $(a, 0)$, and use polar coordinates, putting $m = \tan \alpha$, we get

$$\frac{\left\{r\cos(\theta-\alpha) - \frac{a\cos\alpha}{\sin^2\alpha}(1+\lambda^2)\right\}^2}{(1+\lambda^2)^2} + \frac{\left\{r\sin(\alpha-\theta) - \frac{a(1-\lambda^2)}{\sin\alpha}\right\}^2}{(1-\lambda^2)^2} = \frac{a^2}{\sin^4\alpha},$$

or
$$r^2\left\{\frac{\cos^2(\theta-\alpha)}{(1+\lambda^2)^2} + \frac{\sin^2(\theta-\alpha)}{(1-\lambda^2)^2}\right\} = \frac{2ar}{\sin^2\alpha}\left\{\frac{\cos\alpha\cos(\theta-\alpha)}{1+\lambda^2} - \frac{\sin\alpha\sin(\theta-\alpha)}{(1-\lambda^2)}\right\}$$

that is, $r\left[1+\lambda^4-2\lambda\cos 2^2(\theta-\alpha)\right] = \frac{2a}{\sin^2\alpha}(1-\lambda^4)\left[\cos(2\alpha-\theta)-\lambda^2\cos\theta\right]$.

The radius of curvature at the focus is then $\frac{a}{\sin^2\alpha}\,\frac{(1+\lambda^4-2\lambda^2\cos 2\alpha)}{1-\lambda^4}$.

It would be easy to prove most of the relations between the ellipse and parabola geometrically; thus it is at once obvious that the auxiliary circle of the ellipse touches the directrix of the parabola.

4993. (By R. TUCKER, M.A.)—Focal chords of a parabola are drawn at right angles to each other, and upon them, as diameters, circles are drawn: find (1) the equations to their common tangents; and show that (2) these tangents always touch a circle whose radius is equal to the latus rectum of the parabola; and (3) the common chords of the circles pass through the vertex.

Solution by Prof. WOLSTENHOLME, Prof. EVANS, *and others.*

The equation of a circle on a focal chord as diameter will be, if α be the angle which the chord makes with the axis,

$$(x-2a\cot^2\alpha)^2+(y-2a\cot\alpha)^2 = \left(\frac{2a}{\sin^2\alpha}\right)^2,$$

and the one on the chord at right angles

$$(x-2a\tan^2\alpha)^2+(y+2a\tan\alpha)^2 = \left(\frac{2a}{\cos^2\alpha}\right)^2;$$

hence their common chord is

$$-x(\cot\alpha-\tan\alpha)-y = a(\cot\alpha-\tan\alpha),$$

passing through the point $(-a,\ 0)$ or the vertex.

If $x\cos\theta+y\sin\theta = p$ be a common tangent, we shall have

$$p-2a\cos\theta\cot^2\alpha-2a\sin\theta\cot\alpha = \pm\frac{2a}{\sin^2\alpha},$$

$$p-2a\cos\theta\tan^2\alpha+2a\sin\theta\tan\alpha = \pm\frac{2a}{\cos^2\alpha};$$

whence $\quad -\cos\theta(\cot\alpha-\tan\alpha)-\sin\theta = \pm(\cot\alpha-\tan\alpha),$

$$i.e.,\ \tan\alpha-\cot\alpha = \tan\tfrac{1}{2}\theta \text{ or } -\cot\tfrac{1}{2}\theta';$$

and $p = a\cos\theta(\cot^2\alpha+\tan^2\alpha)+a\sin\theta(\cot\alpha-\tan\alpha)\pm a(2+\cot^2\alpha+\tan^2\alpha)$

$\qquad = a\cos\theta(\tan^2\tfrac{1}{2}\theta+2)-a\sin\theta\tan\tfrac{1}{2}\theta+a(4+\tan^2\tfrac{1}{2}\theta)$

or $\quad = a\cos\theta'(\cot^2\tfrac{1}{2}\theta'+2)+a\sin\theta'\cot\tfrac{1}{2}\theta'-a(4+\cot^2\tfrac{1}{2}\theta'),$

which reduce respectively $4a+2a\cos\theta$, and $-4a+2a\cos\theta'$, or the two common tangents are $(x-2a)\cos\theta+y\sin\theta = 4a$, $(x-2a)\cos\theta'+y\sin\theta'$

$= -4a$, both of which touch the circle $(x-2a)^2+y^2 = (4a)^2$, a circle of radius equal to the latus-rectum and touching the directrix.

The equations of the two common tangents, in terms of α, will be

$$x-6a+2y(\tan\alpha-\cot\alpha)-(x+2a)(\tan\alpha-\cot\alpha)^2 = 0,$$

and $$x-6a+2y(\cot\alpha-\tan\alpha)-(x+2a)(\tan\alpha-\cot\alpha)^2 = 0.$$

5006. (By S. Tebay, B.A.)—A gallon vessel is filled with water, having 1 lb. of salt in solution; and a gallon of water is slowly added to the mixture; find (1) how much salt still remains in the vessel; and (2) how much water must be added that half the salt shall be carried over.

I. *Solution by* Professor Tanner, M.A.

Let $y =$ number of lbs. of salt in the vessel after the addition of x gallons of water. Suppose dx gallons now added. Then $1+dx$ gallons contain y lbs. of salt. But dx gallons overflow, leaving $y+dy$ lbs. of salt in solution in 1 gallon. If we suppose the mixture to be complete, we have

$$\frac{1+dx}{y} = \frac{1}{y+dy};$$

whence, neglecting quantities of second order, we have $dy+ydx = 0$, $y = ce^{-x}$, and from the data, $c = 1$.

Hence (1) if 1 gallon has been added $\left(\text{so that } x=1,\ y=\frac{1}{e}\right)$, $\frac{1}{e}$ lbs. of salt remain; and (2), if $\frac{1}{2}$ lb. of salt remains (so that $y=\frac{1}{2}$, $x=\log_e 2$), $\log_e 2$ gallons of water have been added.

II. *Solution by the* Proposer.

Let a be the weight of water, b the weight of salt, c the weight of water added per second, and x the weight of salt remaining in the vessel at the end of the time t. When t becomes $t+dt$, we have

$$x+\frac{dx}{dt}dt = x-\frac{x}{a}c\,dt;$$

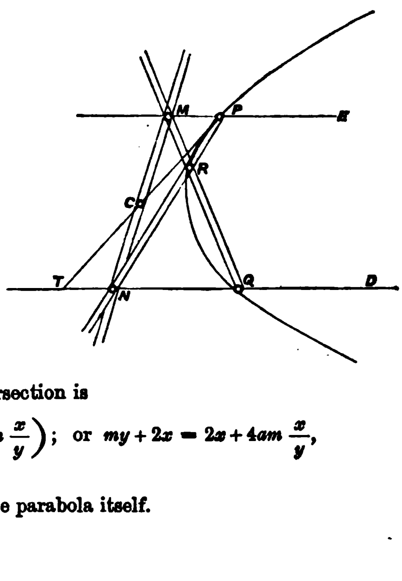
therefore $$\frac{dx}{dt} = -\frac{cx}{a}, \text{ and } x = be^{-\frac{ct}{a}}.$$

Let $ct = a$, and $b = 1$; then we have $x = e^{-1}$.

Let $x = \frac{1}{2}$, then $ct = a\log 2$, or $\log 2$ gals. must be added.

4985. (By S. A. RENSHAW.)—Let P and Q be points on a parabola of which PE, QD are diameters. Draw the tangent PT, and bisect it in C. About C let the arm MCN oscillate, the points M and N moving in the diameters PE, QD produced. Then, if PN, QM be arms revolving about P and Q as centres, show that their intersection R will always move in the parabola.

Solution by the PROPOSER, Prof. WOLSTENHOLME, *and others.*



Referring to the diameter and tangent through P, and taking $y^2=4ax$ as the parabola, the co-ordinates of Q may be taken as $am^2, 2am$; then those of C are $0, am$; and if we take the equation of MCN to be $y-am=\mu x$, then at M, $x=-\dfrac{am}{\mu}$, $y=0$; at N, $x=\dfrac{am}{\mu}$, $y=2am$; and the equations of PN, QM will be $y=2\mu x$, and

$$y=\frac{2}{m+\dfrac{1}{\mu}}\left(x+\frac{am}{\mu}\right);$$

whence the locus of their intersection is

$$y=\frac{2}{m+\dfrac{2x}{y}}\left(x+2am\,\frac{x}{y}\right);\quad\text{or}\quad my+2x=2x+4am\,\frac{x}{y},$$

or $y^2=4ax$, that is to say, the parabola itself.

4915. (By Dr. COLLINS.)—Prove that (1) the equation of the parabola asymptote of

$$ay^{n-2}x^2+a'y^{n-3}+x^3\,a''y^{n-4}x^4+\text{\&c.}+by^{n-1}+b'y^{n-2}x+b''y^{n-3}x^2+\text{\&c.}$$
$$+cy^{n-2}+c'y^{n-3}x+\text{\&c.}=0$$

is
$$ax^2+by+\frac{ab'-a'b}{a}x+c+\frac{b^2(aa''-a'^2)+ab(a'b'-ab'')}{a^3}=0;$$

(2) the parabolic asymptote of third order of

$$ay^{n-3}x^3+a'y^{n-4}x^4+a''y^{n-5}x^5+\text{\&c.}\;+by^{n-1}+b'y^{n-2}x+b''y^{n-3}x^2+\text{\&c.}$$
$$+cy^{n-2}+c'y^{n-3}x+c''y^{n-4}x^2+\text{\&c.}\;dy^{n-3}+d'y^{n-4}x+\text{\&c.}=0$$

is
$$a^3x^3 + a^2by^2 + a(ab'-a'b)xy$$
$$+\left\{a(ab''-a''b)-a'(ab'-a'b)\right\}x^2 + Ay + Bx + C = 0;$$

(3) in the former case, when c is not $= 0$, the asymptotic common parabola above indicated has really a five-pointic contact at infinity with the given curve; and (4) find A, B, C in the second case (which is not done by Euler in pp. 98 and 102 of his *Analysis Infinitorum*) so that the cubic asymptote partly indicated above may really have a nine-point contact at infinity with the given curve of the nth order when b is not $= 0$; for when $b = 0$ in the first equation, the line projected to infinity is not a tangent to the curve, but a straight line passing through a double point of it; and when $b = 0$ in the second equation, then a tangent at a double point (and not at a point of inflection) has been projected to infinity.

Solution by Professor NASH, M.A.

1. The given equation can be written

$$ax^2 + \frac{by + b'x + b''\frac{x^2}{y} + \ldots\ldots + c + c'\frac{x}{y} + \ldots\ldots}{1 + \frac{a'}{a}\frac{x}{y} + \frac{a''}{a}\frac{x^2}{y^2} + \ldots\ldots} = 0,$$

or
$$ax^2 + \left(\left(by + b'x + b''\frac{x^2}{y} + \ldots\ldots + c + c'\frac{x}{y} + \ldots\right)\left(1 - \frac{a'}{a}\frac{x}{y}\right.\right.$$
$$\left.\left. - \frac{a''}{a}\frac{x^2}{y^2} + \frac{a'^2}{a^2}\frac{x^2}{y^2} + \ldots\ldots\right)\right) = 0.$$

If the equation of the asymptotic parabola be $ax^2 + Bx + C + Dy < 0$, if x be infinite and of the first order, y will be of the second order; and since the curve and its asymptote coincide when x is infinite, we must neglect all terms of degrees -1, -2, &c., in x.

Therefore the equation becomes

$$ax^2 + by + b'x + c + \frac{a'b}{a}x + \frac{x^2}{y}\left(b'' - \frac{a'b'}{a} - \frac{a''b}{a} + \frac{a'^2b}{a^2}\right) = 0.$$

Using $y = -\frac{ax^2}{b}$ as a first approximation, and substituting in the last term, we obtain for the equation of the asymptote

$$ax^2 + by + \frac{ab' - a'b}{a}x + c + \frac{b^2(aa'' - a'^2) + ab(a'b' - ab'')}{a^3} = 0.$$

2. As in (1), the given equation can be written

$$ax^3 + \left(by^2 + b'yx + b''x^2 + b'''\frac{x^3}{y} + \ldots\ldots + cy + c'x + \ldots\ldots + d + \ldots\ldots\right)$$
$$\left(1 - \frac{a'}{a}\frac{x}{y} + \frac{a'^2 - aa''}{a^2}\frac{x^2}{y^2} + \frac{2a'a'' - a'''a}{a^2}\frac{x^3}{y^3} + \ldots\ldots\right) = 0.$$

Selecting only the terms of the second degree, we have

$$ax^3 + by^2 + b'yx + b''x^2 - \frac{a'b}{a}xy - \frac{a'b'}{a}x^2 - \frac{a''b}{a}x^2 + \frac{a'^2b}{a^2}x^2 + \frac{Ax + By + C}{a^2} = 0.$$

$$i.e.,\ a^3x^3 + a^2by^2 + a(ab' - a'b)xy + [a(ab'' - a''b) - a'(ab' - a'b)]x^2$$
$$+ Ax + By + C = 0.$$

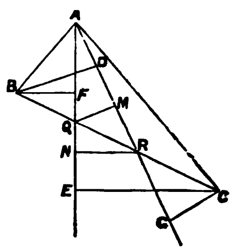
To find the values of A, B, C, we must take, I believe, 59 more terms in the above product, and then approximate by successive substitutions.

4977. (By the Editor.)—If the base BC of a triangle ABC be trisected in Q, R; prove that

$$\sin BAR \sin CAQ = 4 \sin BAQ \sin CAR \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$(\cot BAQ + \cot QAR)(\cot CAR + \cot RAQ) = 4 \operatorname{cosec}^2 QAR \ldots \quad (2).$$

I. *Solution by* T. Merrick, J. O'Regan, A. W. Cave, *and others*.



1. $\operatorname{Sin} BAR \,.\, \sin CAQ = \dfrac{BD}{AB} \cdot \dfrac{CE}{CA} = \dfrac{2QM}{AB} \cdot \dfrac{2RN}{AC}$

$$= 4\frac{CG}{AC} \cdot \frac{BF}{AB} = 4 \sin CAR \,.\, \sin BAQ.$$

2. $(\cot BAQ + \cot QAR)(\cot CAR + \cot RAQ)$

$$= \frac{\sin BAR}{\sin BAQ \,.\, \sin QAR} \cdot \frac{\sin CAQ}{\sin CAR \,.\, \sin RAQ}$$

$$= \frac{4}{\sin^2 QAR} = 4 \operatorname{cosec}^2 QAR.$$

II. *Solution by* Prof. Wolstenholme; R. Tucker, M.A.; *and others*.

1. $\dfrac{\sin BAR \,.\, \sin QAC}{\sin BAQ \sin RAC} = \dfrac{BR \,.\, QC}{BQ \,.\, RC} = 4$ [being the anharmonic range (BRQC) as seen at A],

$$= \frac{\sin(BAQ + QAR) \,.\, \sin(QAR + RAC)}{\sin BAQ \sin RAC}$$

$$= (\cot BAQ + \cot QAR)(\cot QAR + \cot RAC) \,.\, \sin^2 QAR.$$

2. Therefore $(\cot BAQ + \cot QAR)(\cot QAR + \cot RAC) = 4 \operatorname{cosec}^2 QAR.$

III. *Solution by* J. HAMMOND; S. A. RENSHAW; J. W. MULCASTER; *and others.*

1. Since $\triangle BAR = 2\triangle BAQ$, and $\triangle CAQ = 2\triangle CAR$;

therefore $BA \,.\, AR \sin BAR \,.\, CA \,.\, AQ \sin CAQ$
$$= 4BA \,.\, AQ \sin BAQ \,.\, CA \,.\, AR \sin CAR;$$

hence $\sin BAR \sin CAQ = 4 \sin BAQ \sin CAR$.

2. Now $\cot BAQ + \cot QAR = \dfrac{\sin(BAQ+QAR)}{\sin BAQ \sin QAR}$;

hence $(\cot BAQ + \cot QAR)(\cot CAR + \cot RAQ)$
$$= \frac{\sin BAR \sin CAQ}{\sin BAQ \sin CAR \sin^2 QAR} = 4 \operatorname{cosec}^2 QAR \quad \text{.........} \quad (2).$$

4932. (By R. W. GENESE, M.A.)—Show that (1) the equations to the tangents to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, parallel to $y=mx$, are $y-mx = \beta - m\alpha \pm k(bm^2+2hm+a)^{\frac{1}{2}}$, where (α, β) are the coordinates of the centre, and $(ab-h^2)\,k = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$; and (2) the equations to parallel normals are $y-mx = \beta - m\alpha \pm k\dfrac{(m^2-1)\,h+(a-b)\,m}{(bm^2+2hm+a)^{\frac{1}{2}}}$.

Solution by L. W. JONES, B.A.; H. T. GERRANS, B.A.; S. TEBAY, B.A.; *and others.*

Taking the central equation $ax^2+2hxy+by^2+c' = 0$, where
$$c' = c - \frac{2fgh-af^2-bg^2}{ab-h^2},$$
the equation to the tangent is $y-mx = y'-mx'$, where $m = -\dfrac{ax'+hy'}{hx'+by'}$. Combining this with the equation to the curve, we have
$$x' = \pm\frac{k(bm+h)}{(bm^2+2hm+a)^{\frac{1}{2}}},\quad y' = \mp\frac{k(hm+a)}{(bm^2+2hm+a)^{\frac{1}{2}}},$$
where
$$k = \left(-\frac{c'}{ab-h^2}\right)^{\frac{1}{2}};$$
therefore
$$y-mx = \pm k(bm^2+2hm+a)^{\frac{1}{2}}.$$

Similarly, for the normal, putting $m = \dfrac{hx' + by'}{ax' + hy'}$, we find

$$x' = \pm \frac{k(hm-b)}{(am^2-2hm+b)^{\frac{1}{2}}}, \qquad y' = \pm \frac{k(h-ma)}{(am^2-2hm+b)^{\frac{1}{2}}};$$

and
$$y-mx = \pm k \frac{(m^2-1)h + (a-b)m}{(am^2-2hm+b)^{\frac{1}{2}}}.$$

5020. (By W. S. B. Woolhouse, F.R.A.S.)—Let $1, \delta_1, \delta_2, \delta_3 \ldots \delta_n$ be the first differences of the coefficients of the expansion of the binomial $(1+x)^{2n}$ taken as far as the central or maximum coefficient; also let

$$\nu = \tfrac{1}{2}(n+1)n, \quad \nu' = \tfrac{1}{2}n(n-1), \quad \nu'' = \tfrac{1}{2}(n-1)(n-2), \text{ \&c.};$$

then show that the algebraic function

$$x^{\nu} - \delta_1 x^{\nu'} + \delta_2 x^{\nu''} - \delta_3 x^{\nu'''} + \text{\&c.}$$

is divisible by $(x-1)^n$ without a remainder; and that the sum of the numerical coefficients of the quotient is equal to $1 \,.\, 3 \,.\, 5 \ldots 2n-1$.

[See Solution to Question 1894, *Reprint*, Vol. V., page 113.]

I. *Solution by* Professor Cayley.

Mr. Woolhouse's elegant theorem depends ultimately on the property of triangular numbers $[\phi(n) = \frac{1}{2}(n^2-n)$, then$]$ $\phi(n+1) = \phi(-n)$; so that, writing down the series of triangular numbers backwards and forwards,

$$\begin{array}{cccccccccccc} \ldots & 10, & 6, & 3, & 1, & 0, & 0, & 1, & 3, & 6, & 10 & \ldots \\ \ldots & & a, & b, & c, & d, & e, & f, & g, & h & & \ldots \end{array}$$

we have, in fact, a continuous single series obtained by giving to n the different negative and positive integer values, zero included.

Thus a particular case is

$$(1+x)^6 - 5(1+x)^3 + 9(1+x) - 5 \equiv 0 \ (\text{mod. } x^3) = 1 \,.\, 3 \,.\, 5x^3 + \text{\&c. } x^4 \ldots,$$

where, on the left-hand side, the exponents are the triangular numbers $\phi(n+1)$, $n = 0$ to 3; and the coefficients after the first are the differences of the binomial coefficients of the power $2n$ $(n=3)$; viz., the binomial coefficients being $1, 6, 15, 20, 15, 6, 1$,

the differences taken as far as they are positive are

$$5, \ 9, \ 5.$$

Expanding the several terms and writing down only the coefficients, we have a diagram

$$\begin{array}{r|l} & 1, \ 6, \ 15, \ 20, \ 15, \ 6, \ 1 \\ -5 & 1, \ 3, \ \ 3, \ \ 1 \\ +9 & 1, \ 1 \\ -5 & 1 \end{array}$$

and the theorem in the particular case depends on the identities

$$\begin{aligned} 1 - 5 + 9 - 5 &= 0, \\ 6 - 15 + 9 &= 0, \\ 5 - 15 &= 0, \\ 20 - 5 &= 1 \,.\, 3 \,.\, 5; \end{aligned}$$

and writing, as above, h, g, f, e to denote the triangular numbers 6, 3, 1, 0, these may be replaced by

$$\begin{array}{llll} h^0 & -5\,g^0 & +9f^0-5e^0 & =0, \\ h & -5\,g & +9f-5e & =0, \\ \tfrac{1}{2}h\,(h-1) & -5\,.\,\tfrac{1}{2}g\,(g-1) & +\ldots & =0, \\ \tfrac{1}{6}h\,(h-1)\,(h-2) & -5\,.\,\tfrac{1}{6}g\,(g-1)\,(g-2) & +\ldots & =1\,.\,3\,.\,5\,; \end{array}$$

or, reducing each equation by those which precede it, these become

$$\begin{aligned} h^0-5g^0+9f^0-5e^0 &= 0, \\ h^1-5g^1+9f^1-5e^1 &= 0, \\ h^2-5g^2+9f^2-5e^2 &= 0, \\ h^3-5g^3+9f^3-5e^3 &= 1\,.\,2\,.\,3\,.\,1\,.\,3\,.\,5. \end{aligned}$$

Consider any one of these, for instance the third; the function on the left-hand is $\quad 1h^2-(6-1)\,g^2+(15-6)f^2-(20-15)\,e^2$,
or, introducing the values b, c, d as above,

$$1h^2-6g^2+15f^2-20e^2+15d^2-6c^2+1b^2,$$

which is, in fact, $=0$ if b, c, d, e, f, g, h are *any* successive triangular numbers; viz., this is an immediate consequence of the well-known theorem

$$1\,(\theta+6)^m-6\,(\theta+5)^m+15\,(\theta+4)^m-20\,(\theta+3)^m+15\,(\theta+2)^m-6\,(\theta+1)^m+\theta^m$$
$$=\Delta^6\,\theta^m, =0 \text{ for any value of } m \text{ up to } m=5\,; \text{ and}$$
$$=1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6 \text{ for } m=6.$$

We have thus all the equations except the last; and as regards the last equation, observe that the equation to be verified is

$$1\,[\tfrac{1}{2}\,(\theta+6)\,(\theta+5)]^3-6\,[\tfrac{1}{2}\,(\theta+5)\,(\theta+4)]^3+\ldots=1\,.\,2\,.\,3\,.\,1\,.\,3\,.\,5,$$

viz., this may be replaced by

$$1\,(\theta+6)^6-6\,(\theta+5)^6+\ldots=2^3\,.\,1\,.\,2\,.\,3\,.\,1\,.\,3\,.\,5=2\,.\,4\,.\,6\,.\,1\,.\,3\,.\,5$$
which is right. $\qquad\qquad\qquad\qquad =1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6.$

It is clear that the proof, although worked out on a particular case, is perfectly general; and Mr. Woolhouse's theorem is thus proved.

II. *Solution by the* PROPOSER.

According to the Question, the series of values of the exponent ν are

$$\nu=\frac{(n+1)\,n}{2},\quad \nu'=\frac{n\,(n-1)}{2},\quad \nu''=\frac{(n-1)\,(n-2)}{2},\quad \ldots\ldots,$$

and, generally, $\qquad \nu^{(h)}=\dfrac{(n+1-h)\,(n-h)}{2}.$

These values diminish and merge into zero at $\nu^{(n)}$. If, however, the series be extended, and succeeding values of ν be taken as far as $\nu^{(2n+1)}$, they will exhibit a recurrence of the same set of numbers, only in a reverse order. This is shown by the identity $\nu^{(h)}=\nu^{(k)}$ when $h+k=2n+1$. It will here be understood that the symbols placed in parentheses do not indicate powers, but simply denote ordinal numbers, or accents, showing the positions of the particular values of ν in the series.

Let 1, β_1, β_2, β_3, … β_2, β_1, 1 be the entire set of numerical coefficients of the expansion of the binomial $(1+x)^{2n}$; and let us consider the algebraic function

$$x^{\nu}-\beta_1x^{\nu'}+\beta_2x^{\nu''}-\beta_3x^{\nu'''}\ldots\ldots+\beta_2x^{\nu^{(2n-2)}},\ -\beta_1x^{\nu^{(2n-1)}},\ +x^{\nu^{(2n)}}\ \ldots..(1).$$

According to the law of identities just mentioned, the concluding terms on the right hand of (1), taken in reverse order, are equivalent to

$$x^{\nu'} - \beta_1 x^{\nu''} + \beta_2 x^{\nu'''} \ldots\ldots ;$$

and, by the substitution of these, (1) becomes

$$x^{\nu} - \beta_1 x^{\nu'} + \beta_2 x^{\nu''} - \beta_3 x^{\nu'''} \ldots\ldots + x^{\nu'} - \beta_1 x^{\nu''} + \beta_2 x^{\nu'''} \ldots\ldots$$
$$= x^{\nu} - \delta_1 x^{\nu'} + \delta_2 x^{\nu''} - \delta_3 x^{\nu'''} \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which is the function stated in the proposed theorem. In the primitive form (1), since the coefficients are those of the development of $(1-x)^{2n}$, it expresses the value of a difference of the $2n$th order of the $2n+1$ quantities $\quad x^{\nu}, x^{\nu'}, x^{\nu''}, x^{\nu'''} \ldots\ldots x^{\nu^{(2n)}}$.

If these last be written

$$\{1+(x-1)\}^{\nu}, \ \{1+(x-1)\}^{\nu'}, \ \{1+(x-1)\}^{\nu''}, \ \ldots\ldots \ \{1+(x-1)\}^{\nu^{(2n)}},$$

and the several binomials expanded in powers of $(x-1)$, the coefficient of $(x-1)^n$ in each of them will be of the form $\frac{\nu(\nu-1)\ldots}{\lfloor n}$, an algebraic function of ν, in which the term involving the highest power is $\frac{\nu^n}{\lfloor n}$. Thus, with reference to the general value $\{1+(x-1)\}^{\nu^{(h)}}$, having $\nu^{(h)} = \frac{(n+1-h)(n-h)}{2}$, the coefficient of $(x-1)^n$ consists of terms involving powers of h, of which the highest is $\frac{1}{\lfloor n}\left(\frac{h^2}{2}\right)^n = \frac{h^{2n}}{2^n \lfloor n}$. Now, for consecutive values of the number h, the difference of the $2n$th order with respect to corresponding values of the power h^{2n} is constant and equal to $\lfloor 2n$; and the same order of difference vanishes for all powers of h inferior to h^{2n}, and consequently for the coefficients of all powers of $(x-1)$ inferior to $(x-1)^n$. Hence $(x-1)^n$ is the lowest power that can appear in the $2n$th difference; and for it the coefficient is $\frac{\lfloor 2n}{2^n \lfloor n}$ or $1.3.5 \ldots 2n-1$. The proposed function (2), or its equivalent (1), is thus reduced to the form

$$(1.3.5 \ldots 2n-1)(x-1)^n + C(x-1)^{n+1} \ldots\ldots + (x-1)^{\nu} \ldots\ldots\ldots (3).$$

And the final result, after division by $(x-1)^n$, is of the form

$$1.3.5 \ldots 2n-1 + C(x-1) \ldots\ldots + (x-1)^{\nu'} \ldots\ldots\ldots\ldots\ldots (\text{I.}),$$

in which, after expansion, the sum of the coefficients of the several powers of x, being the value obtained on putting *unity* for x, is of course $1.3.5 \ldots 2n-1$.

From what precedes it is evident that the theorem may be generalized thus:—If the law of the formation of the series of values of the exponent ν be such that the mth difference $\Delta_m \nu$ is constant and equal to $c \lfloor m$; then will the mnth difference $\Delta_{mn} x^{\nu}$ be divisible by $(x-1)^n$ without a remainder, and the sum of the numerical coefficients of the quotient will be equal to $\frac{\lfloor mn}{\lfloor n} c^n$.

The chief interest attached to the theorem proposed in the foregoing Question is the remarkable property of the algebraic quotient (I.) as an intersectional function, appertaining to certain stated geometrical constructions.

If there be $2n$ points which, if consecutively joined by straight lines, would form the contour of an irregular convex polygon, they may be united in pairs by n lines in $1 \,.\, 3 \,.\, 5 \ldots 2n-1$ ways, giving as many diagrams, and these diagrams may be classified according to the number of intersections existing amongst the n lines. The interesting property here referred to is, that the numerical coefficient of any power x^k in the function (I.) will invariably show the number of diagrams having k intersections.

4995. (By Prof. Townsend, F.R.S.)—Two tangents being supposed drawn to a plane nodal cubic from a point on the curve, shew that, as the point varies in position on the curve :—

(a) The chord of contact of the tangents envelopes a conic, which touches the two nodal tangents at their two points of intersection with the inflexional axis of the cubic.

(b) The enveloped conic has also triple contact with the cubic itself, the three points of contact being those of the three tangents to it from its three points of inflexion.

I. *Solution by* J. J. Walker, M.A.

An analytical Solution of this Question, though not, perhaps, as short as the geometrical, may have some interest. The equation to the cubic referred to its nodal tangents and the axis of inflexion (z) is

$$ax^3 + by^3 + 6dxyz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

the polar conic of $x'y'z'$ is

$$ax'x^2 + bx'y^2 + 2dz'xy + 2dx'yz + 2dy'zx = 0 \quad \ldots\ldots\ldots\ldots \text{(2)};$$

and if $x'y'z'$ is a point on (1), $ax'^3 + by'^3 + 6dx'y'z' = 0 \ldots\ldots\ldots\ldots \text{(3)}.$

Multiplying (1) by $x'y'(y'x + x'y)$, (2) by $-3x'y'xy$, (3) by x^2y^2, and adding, the result is $(y'x - x'y)^2 (ax'x^2 + by'y^2) = 0 \ldots\ldots\ldots\ldots \text{(4)}.$

Discarding the first factor, from the other, multiplied by $y'x + x'y$, subtract (1) multiplied by $x'y'$; then we have

$$xy\,(ax'^2x + by'^2y - 6dx'y'z) = 0 \ldots\ldots\ldots\ldots \text{(5)},$$

the third factor in which is evidently the chord of contact of tangents from $x'y'z'$ to the cubic, and envelopes the conic

$$abxy - 9d^2z^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}.$$

One of the three tangents from the points of inflexion is

$$a^{\frac{2}{3}} b^{\frac{1}{3}} x + a^{\frac{1}{3}} b^{\frac{2}{3}} y + 6dz = 0,$$

since this line meets the cubic at the points determined by

$$(a^{\frac{1}{3}} x + b^{\frac{1}{3}} y)\,(a^{\frac{1}{3}} x - b^{\frac{1}{3}} y)^2 = 0;$$

but it also meets the conic (6) at the coincident points determined by the same square factor.

II. *Solution by the* PROPOSER.

By reciprocation the property becomes evidently transformed into the following: viz., a tangent chord, drawn arbitrarily in a plane tricuspidal quartic curve, being supposed to change position in the curve, the tangents at its extremities intersect, in every position, on a conic passing through the two extremities of the bitangent chord, and having triple contact with the quartic at its three points of intersection with its three cuspidal tangents; which was proved, in both its parts, in an answer to a former Question proposed by Professor WOLSTENHOLME. [See *Reprint*, Vol. XX., Questions 4120 and 4142.]

4978. (By R. MOON, M.A.)—Prove (1) that, when a small disturbance is propagated through a cylindrical tube filled with air, the motion being wholly parallel to the axis, we shall have

$$0 = \int_x \left(\frac{v^2}{a^2} + \frac{\rho^2}{D^2}\right) + 2a \int_t \left(\frac{v}{a} \cdot \frac{\rho}{D}\right),$$

where v, ρ are the velocity and density at the time t of a particle, the ordinate of whose point of rest is x, and a and D are respectively the velocity of propagation and mean density; and (2) write down the corresponding formula when the accurate equations of motion are employed.

Solution by the PROPOSER.

1. If y be the ordinate of the particle at the time t, the accurate equation of motion will be $\quad \dfrac{d^2y}{dt^2} = a^2 \left(\dfrac{dy}{dx}\right)^2 \cdot \dfrac{d^2y}{dx^2};$

or, since $\dfrac{D}{\rho} = \dfrac{dy}{dx}$, $\qquad 0 = \dfrac{dv}{dt} + \dfrac{a^2}{D} \cdot \dfrac{d\rho}{dx}$ ....(1).

Also, since $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{dy}{dt}\right)$, $\quad 0 = \dfrac{v}{dx} \pm \dfrac{D}{\rho^2} \cdot \dfrac{d\rho}{dt}$ ....(2).

We may take (1) and (2) for the accurate equations of motion, the latter of which becomes, when the motions are small,

$$0 = \frac{dv}{dx} + \frac{1}{D} \cdot \frac{d\rho}{dt} \quad\text{..................................(3).}$$

Multiplying (1) by $\dfrac{v}{a^2}$, and (3) by $\dfrac{\rho}{D}$, we get

$$0 = \frac{v}{a^2}\left(\frac{dv}{dt} + \frac{a^2}{D} \cdot \frac{d\rho}{dx}\right) + \frac{\rho}{D}\left(\frac{dv}{dx} + \frac{1}{D} \cdot \frac{d\rho}{dt}\right)$$

$$= \frac{v}{a^2} \cdot \frac{dv}{dt} + \frac{\rho}{D^2} \cdot \frac{d\rho}{dt} + \frac{\rho}{D} \cdot \frac{dv}{dx} + \frac{v}{D} \cdot \frac{d\rho}{dx};$$

whence, integrating with respect to x and t, we get

$$0 = \int_x \left(\frac{v^2}{a^2} + \frac{\rho^2}{D^2}\right) + 2a \int_t \left(\frac{v}{a} \cdot \frac{\rho}{D}\right),$$

or, as the equation may be more simply and significantly written,

$$0 = \int_x \left(\frac{v^2}{a^2} + \frac{\rho^2}{D^2}\right) + \int_{at}\left(2\,\frac{v}{a}\cdot\frac{\rho}{D}\right).$$

2. Multiplying (1) by v, and integrating with respect to t, we get

$$0 = \tfrac{1}{2}v^2 + \frac{a^2}{D}\int_t v\,\frac{d\rho}{dx},$$

or, integrating with respect to x,

$$0 = \int_x \tfrac{1}{2}v^2 + \frac{a^2}{D}\int_t \left(v\rho - \int_x \rho\,\frac{dv}{dx}\right) = \int_x \tfrac{1}{2}v^2 + \frac{a^2}{D}\int_t \left(v\rho + \int_x \frac{D}{\rho}\cdot\frac{d\rho}{dt}\right),$$

in virtue of equation (2); or

$$0 = \int_x \tfrac{1}{2}v^2 + \frac{a^2}{D}\int_t v\rho + a^2\int_x \log_e \rho = \int_x \left(\frac{v^2}{a^2} + \log_e \rho^2\right) + 2a\int \left(\frac{v}{a}\cdot\frac{\rho}{D}\right).$$

The presence of the symbol of integration with respect to t in this last equation involving the addition of an arbitrary function of x, the equation may be written

$$0 = \int_x \left(\frac{v^2}{a^2} + \log_e \frac{\rho^2}{D^2}\right) + 2a\int_t \left(\frac{v}{a}\cdot\frac{\rho}{D}\right),$$

or

$$0 = \int_x \left(\frac{v^2}{a^2} + \log_e \frac{\rho^2}{D^2}\right) + \int_{at}\left(2\,\frac{v}{a}\cdot\frac{\rho}{D}\right).$$

4419. (By the EDITOR.)—Let $V_1, V_2, \ldots, V_n$ be conical solids, which have a common vertex and their bases on n plane areas anyhow situated in space; then, if $a_1, a_2, \ldots, a_n$ be given numbers, and C a constant volume, find (1) the locus of the common vertex when

$$a_1V_1 + a_2V_2 + \ldots + a_nV_n = C,$$

and (2) the analogous locus in a plane, when $V_1, V_2, \ldots, V_n$ are triangles whose bases are given straight lines.

I. *Solution by the* Rev. J. R. WILSON, M.A.

1. Let $l_1x + m_1y + n_1z = p_1,\ l_2x + m_2y + n_2z = p_2,\ \ldots\ l_nx + m_ny + n_nz = p_n$ be the equations to the n planes, and $A_1, A_2, \ldots A_n$ their areas. If x, y, z be the common vertex, the volumes of the cones are

$$\tfrac{1}{3}A_1(p_1 - l_1x - m_1y - n_1z),\quad \tfrac{1}{3}A_2(p_2 - l_2x - m_2y - n_2z),\quad \ldots\ldots$$
$$\ldots\ldots\quad \tfrac{1}{3}A_n(p_n - l_nx - m_ny - n_nz).$$

Hence, expressing the equality in the question, and collecting the coefficients, we have for the locus the plane

$$(a_1A_1l_1 + a_2A_2l_2 + \ldots + a_nA_nl_n)\,x + (a_1A_1m_1 + a_2A_2m_2 + \ldots + a_nA_nm_n)\,y$$
$$+ (a_1A_1n_1 + a_2A_2n_2 + \ldots + a_nA_nn_n)\,z$$
$$= a_1A_1p_1 + a_2A_2p_2 + \ldots + a_nA_np_n - 3C.$$

2. Let $x\cos\alpha_1 + y\sin\alpha_1 = p_1,\quad x\cos\alpha_2 + y\sin\alpha_2 = {}_2,\quad \ldots\ldots$ $\ldots\ldots\ x\cos\alpha_n + y\sin\alpha_n = p_n$ be the equations to the n straight lines,

and c_1, c_2, c_3 their lengths. Then the areas of the triangles are

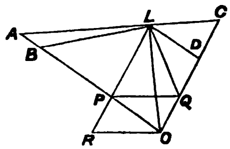
$$\tfrac{1}{2}c_1(p_1 - x\cos\alpha_1 - y\sin\alpha_1),\quad \tfrac{1}{2}c_2(p_2 - x\cos\alpha_2 - y\sin\alpha_2),\quad \ldots\ldots$$
$$\tfrac{1}{2}c_n(p_n - x\cos\alpha_n - y\sin\alpha_n).$$

Hence the locus in this case is the straight line

$$(a_1c_1\cos\alpha_1 + a_2c_2\cos\alpha_2 + \ldots + a_nc_n\cos\alpha_n)\,x$$
$$+ (a_1c_1\sin\alpha_1 + a_2c_2\sin\alpha_2 + \ldots + a_nc_n\sin\alpha_n)\,y$$
$$= a_1c_1p_1 + a_2c_2p_2 + \ldots + a_nc_np_n - 2C.$$

II. *Solution by* HENRY STANLEY MONCK.

As the case of the solid figures and the case of the triangles rest upon the same principles, I take the latter as the simpler, and proceed to show that the locus in question is a straight line. Of course, in the other case it will be a plane.



Let AB, CD be the given bases (I first take the case of two only), and suppose L to be a point such that $aV + bV'$ = a constant. Let AB, CD meet at O. Take $OP = a.AB$ and $OQ = b.CD$; join PQ, and draw OR equal and parallel to PQ. Then the locus is a straight line through L parallel to PQ. Join all the points to L. Then $\Delta OPL = aV$, $\Delta OQL = bV'$, and the quadrilateral OPQL $= aV + bV'$ = a constant. But ΔPQL = quadrilateral OPQL $- \Delta OPQ = aV + bV' - \Delta OPQ$ = constant, and therefore the locus of L is a straight line parallel to PQ or to OR. Now since this line is parallel to OR, the triangle whose base is OR, and whose vertex is on this line, is always of the same area. To find its area, let us suppose L to be the point where the line PR, which is parallel to OC, meets the locus-line. Then
$$\Delta ORL = \Delta ORP + \Delta OPL = \Delta OPQ + \Delta OPL$$
$$= \Delta OLQ + \Delta OPL = aV + bV',$$
and since a parallel to PQ may be drawn through any point, this result is universally true. Now if we take in a third base EF, and inquire the locus for $aV + bV' + cV''$ = constant, and if we denote the area of the triangle standing on OR by v, the question reduces to this—find the locus for $v + cV''$ = constant; which of course can be found as before. The same course may be pursued with any number of bases.

4999. (By the Rev. C. TAYLOR, M.A.)—A polygon circumscribes a fixed ellipse and each of its vertices moves on a fixed confocal. Through each vertex an ellipse is described with the adjacent points of contact as foci: show that the areas of their minor auxiliary circles are connected by a linear relation.

Solution by the PROPOSER.

The latus rectum of each of the variable ellipses is constant. (See *Messenger of Mathematics*, Vol. IV., p. 17.)

Let α_1, β_1; α_2, β_2, &c. be the semi-axes of the variable ellipses.

Then we have $\qquad \dfrac{\beta_1^2}{\alpha_1} = c_1,\quad \dfrac{\beta_2^2}{\alpha_2} = c_2,\quad$ &c.;

therefore $\dfrac{\beta_1^2}{c_1}+\dfrac{\beta_2^2}{c_2}+\&c. = \alpha_1+\alpha_2+\&c. = \frac{1}{2}$ perimeter of polygon
$= $ a constant (*Messenger*, Vol. IV., p. 21);

therefore $\pi\beta_1^2$, $\pi\beta_2^2$, &c. are connected by a linear relation.

When the polygon degenerates into a triangle where three vertices move on the same confocal, then $c_1=c_2=c_3$, and

$$\pi\beta_1^2+\pi\beta_2^2+\pi\beta_3^2 = \text{a constant},$$

as the Rev. F. D. Thomson has shown independently.

4975. (By Prof. Crofton, F.R.S.)—Three straight lines are drawn at random across a given circle; find the probability of their three intersections lying within the circle.

Solution by St. John Stephen.

Let us first of all consider the probability of the intersection of any two of the lines lying within the circle.

If we take any four points at random in a circle; we can join these points by three pairs of lines. Two pairs will certainly intersect without the circle. The probability, therefore, that the intersection of any two lines drawn at random should lie within the circle $= \frac{1}{3}$. Hence, if three straight lines are drawn at random across a given circle, the probability of their intersections lying within the circle $= (\frac{1}{3})^3 = \frac{1}{27}$.

[On pp. 196, 197 of Prof. Crofton's interesting paper *On Local Probability*, printed in the *Philosophical Transactions* (of the Royal Society) for 1868, a totally different view is taken of this problem, and the result obtained is $\dfrac{4}{\pi^2}-\frac{1}{4}$.]

4600. (By Artemas Martin.)—What condition must a and b satisfy in order that the roots of the cubic equation $x^3+ax-b=0$ may be the sides of a triangle?

Solution by J. J. Walker, M.A.

Such a question can only be discussed on the supposition that the roots of the equation are all positive numbers, which those of the form $x^3+ax-b=0$ cannot be. Consider the equation

$$x^3-ax^2+bx-c=0,$$

where a, b, c are essentially positive. Then, besides the well-known conditions for the roots (α, β, γ suppose) being real, we must have $(\alpha+\beta-\gamma)(\gamma+\alpha-\beta)(\beta+\gamma-\alpha)$ positive, that is, $\Sigma\alpha^2\beta-\Sigma\alpha^3-2\alpha\beta\gamma$ positive; or $ab-3c-(a^3-3ab+3c)-2c$ positive, that is, $-a^3+4ab-8c$ positive; or $4ab > a^3+8c$.

5010. (By G. S. CARR.)—If from any point P on an ellipse two straight lines be drawn terminated by the conjugate diameter and touching a confocal ellipse, prove that they will be equal and constant in length.

Solution by L. W. JONES; E. RUTTER; *and others.*

If p be the length of the perpendicular from the centre on the tangent at P, and θ the angle between the two tangents from P to the second ellipse, each of the lines in question has the length $p \sec \frac{1}{2}\theta$. (See Besant's *Conics*, p. 60.)

The equation to the pair of tangents from P $(a\cos\phi,\ b\sin\phi)$ to the ellipse $\dfrac{x^2}{a^2-\lambda^2}+\dfrac{y^2}{b^2-\lambda^2}=1$ is

$$\left(\frac{ax\cos\phi}{a^2-\lambda^2}+\frac{by\sin\phi}{b^2-\lambda^2}-1\right)^2$$
$$=\left(\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}-1\right)\left(\frac{a^2\cos^2\phi}{a^2-\lambda^2}+\frac{b^2\sin^2\phi}{b^2-\lambda^2}-1\right).$$

The angle θ which these lines contain will be the same as that contained by the parallels through the origin, that is,

$$x^2(\lambda^2-b^2\cos^2\phi)+y^2(\lambda^2-a^2\sin^2\phi)-2ab\,xy\cos\phi\sin\phi=0,$$

therefore $$\tan\theta=\frac{2\left\{a^2b^2\cos^2\phi\sin^2\phi-(\lambda^2-b^2\cos^2\phi)(\lambda^2-a^2\sin^2\phi)\right\}^{\frac{1}{2}}}{2\lambda^2-a^2\sin^2\phi-b^2\cos^2\phi}$$
$$=\frac{2\lambda\,(a^2\sin^2\phi+b^2\cos^2\phi-\lambda^2)^{\frac{1}{2}}}{2\lambda^2-a^2\sin^2\phi-b^2\cos^2\phi};$$

whence we obtain $$p^2\sec^2\tfrac{1}{2}\theta=\frac{a^2b^2}{\lambda^2}.$$

Thus each of the two lines in question has the length $ab\lambda^{-1}$, which is the same for all positions of P.

[This property is proved also in an article by the Rev. C. Taylor, on p. 185 of Vol. III. of the *Messenger of Mathematics.*]

5029. (By R. W. GENESE, M.A.)—Prove that, if the equation $x^3-3px+q=0$ has (1) only one root α real, then $\alpha^2>4p$; and (2), if all three roots be real, any one must be numerically less than $2p^{\frac{1}{2}}$.

Solution by Professor TANNER, M.A.

Since α is a root of $\qquad x^3-3px+q=0,$

we may consider q to be given by the relation

$$\alpha^3-3p\alpha+q=0,$$

so that the given equation may be written

$$x^3-3px-\alpha^3+3p\alpha=0,\quad\text{or}\quad (x-\alpha)(x^2+\alpha x+\alpha^2-3p)=0.$$

Hence the roots other than α are imaginary or real according as

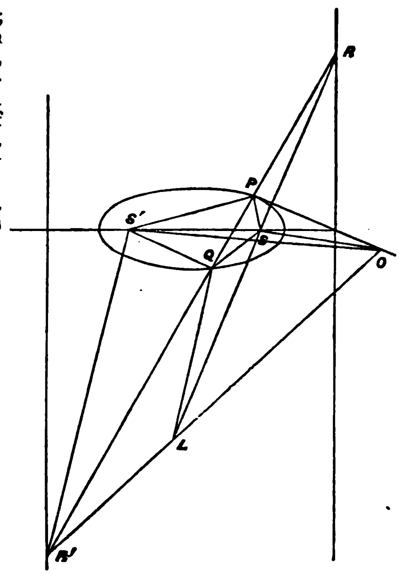
$$\alpha^2-4(\alpha^2-3p) < \text{ or } > 0, \quad \text{that is,} \quad \text{as } -\alpha^2+4p < \text{ or } > 0,$$

that is, as $4p < \text{ or } > \alpha^2$.

Therefore if only one root α be real we have $\alpha^2 > 4p$. If all be real, none can be greater (numerically) than $2p^{\frac{1}{2}}$, since in this case α may be put for either of the roots.

4988. (By the Rev. C. TAYLOR, M.A.)—If a normal meet the conic again in Q, and the directrices in R, R′; and if O be the pole of the chord, and S, S′ the foci; prove that SR, OR′, and S′R′, OR intersect on the normal at Q.

I. *Solution by* Professor WOLSTENHOLME.



Draw the tangents OP, OQ; let SR, OR′ meet in L; join OS, OS′, LQ, SP, SQ, S′P, S′Q. Then, since R is the intersection of the polar of S and the polar of O, R is the pole of OS, and the angle OSR is a right angle, and similarly OS′R′.

Also PSQ − PS′Q = 2 ∠POQ (for any chord PQ that crosses the axis between the foci); therefore

∠PSQ = PS′Q + 2POQ
= 2PS′O + 2POQ

(since OS′ bisects PS′Q)

= 2PR′O + 2POQ

(because a circle goes round R′S′PQ)

= π − 2POR′ + 2POQ
= π − 2QOR′
= π − 2QOL.

But PSQ = π − 2QSL, because OS bisects the angle PSQ. Hence, finally, QOL = QSL, and a circle goes round LQSO; therefore LQO = LSO = a right angle, or LQ is the normal at Q. Similarly, if S′R′, OR meet in L′, L′Q is normal at Q.

II. *Solution by* Professor WILLIAMSON, M.A.

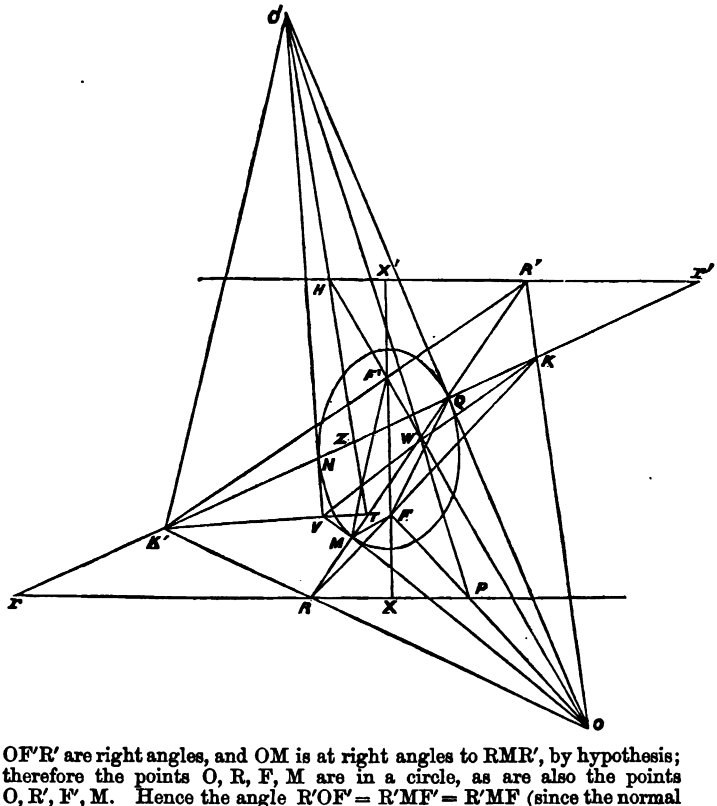
In Professor Wolstenholme's diagram suppose the line OR to be drawn; then it appears immediately, as in the preceding demonstration, that the

angle OSR is a right angle, as also the angle OS′R′. Accordingly (since RPSO is inscribable in a circle) ∠SOR = QPS = QPS′ = S′OR′. Hence, taking away from each the equal angles QOS′ and POS, we have ∠POR = QOR′.

Again, ∠POR = PSR = QSL (since OS bisects the angle PSQ); therefore QSL = QOL, and hence a circle passes round LQSO; therefore LQO = LSO = a right angle, or LQ is the normal at Q.

III. *Solution by* S. A. Renshaw, *and others.*

Complete the figure, and take the ellipse for example. Then, since RR′ is the chord of contact of the tangents OM, OQ, the angles OFR,



OF′R′ are right angles, and OM is at right angles to RMR′, by hypothesis; therefore the points O, R, F, M are in a circle, as are also the points O, R′, F′, M. Hence the angle R′OF′ = R′MF′ = R′MF (since the normal makes equal angles with the focal distances). But ∠R′MF = ROF (by Euc. III. 22); therefore R′OF′ = ROF; and ∠FOM = F′OQ, by a well-known theorem, therefore ∠MOR = R′OQ. Also, since the tangents subtend equal angles at the foci, the angle OFM = OFQ; therefore,

taking from each of these a right angle, viz., OFR or OFK, we have $\angle$MFR = QFK. But MFR = MOR = QOR′, as has been proved; therefore QOR = QFK; hence RF produced and the normal at Q must meet OR′ in the same point; for if not, let RF meet OR′ in K, and join QK; then, since K, Q, F, O are in a circle, the angle KQO = KFO; but KFO is a right angle, therefore KQO is a right angle, which is absurd; therefore RF produced and the normal at Q cannot but meet on OR′; that is, they do meet upon it. Similarly, R′F′ produced and the same normal meet on OR, as may, of course, be shown in the same manner.

Corollaries.—If OF meet the directrix X in P, and O′P be joined, and also OF′ be joined and produced to meet the directrix X′, &c., then OF′ and OP meet on RR′; R′, K, W being respectively the poles of OF′, O′P, OR′. The points T, V, K′ are in a straight line, as are also the points V, W, K, being in each case the intersections of homologous rays of harmonic pencils that have a common ray; and as K was shown to be the pole of HO, so K′ is the pole of THO; also Z is the pole of K′O′, and T that of ORK′.

IV. *Solution by* R. Tucker, M.A.

Using Professor Wolstenholme's figure, as given in Solution I., and bearing in mind that circles go round R′S′PO, RPSO,

we have $\angle$SOQ = $\angle$S′OP (Besant's *Conics*, p. 63).

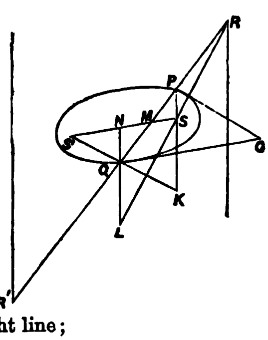
Also $\angle$S′OR′ = $\angle$S′PR′ = $\angle$SPQ = $\angle$SOR,

therefore $\angle$SOR′ = $\angle$S′OR.

Hence $\angle$QOR′ = $\angle$POR = $\angle$PSR = $\angle$QSL.

And Q, S, O, L are concyclic; therefore &c.

V. *Solution by the* Rev. F. D. Thomson, M.A.



Let the normals at P and Q meet SS′ in M and N;

then $\{S'NMS\} = \{R'QPR\} = \{PRR'Q\}$.

Since S′M : MS = S′P : PS = R′P : PR, &c.,

therefore Q $\{S'NMS\}$ = S $\{PRR'Q\}$;

therefore K, L, R′ are in a straight line.

Again, Q $\{S'NSO\}$ = O $\{RPSQ\}$, since each is harmonic; hence we have

Q $\{S'NSO\}$ = S $\{RPOQ\}$ = S $\{PRQO\}$;

therefore K, L, O are in a straight line;

therefore K, L, R′, O are in a straight line.

VI. *Solution by* J. J. WALKER, M.A.

The equation to the ellipse being $b^2x^2+a^2y^2=a^2b^2$, and c^2 being equal to a^2-b^2, the equation to the normal at Q is

$$\frac{a^2\eta x}{b^2-\eta^2}+\frac{b^2\xi y}{a^2-\xi^2}=\frac{c^4\xi\eta}{b^2\xi^2+a^2\eta^2} \text{ (see Quest. 5095)...(1),}$$

the equation to (SR) is $(c-\xi)\,x-\eta y=c\,(c-\xi)$ .... (2),
and the equation to (OR′) is

$$\left\{b^2(c+\xi)-c\eta^2\right\}x+(a^2+c\xi)\,\eta y=b^2(c+\xi)\,\xi+a^2\eta^2 \text{ ....(3).}$$

Multiplying (1) by $(a^2-\xi^2+b^2-\eta^2)\,c\eta\,(b^2-\eta^2)\,(a^2-\xi^2)$,

also (2) by $b^2\,(b^2-\eta^2)\left\{(b^2-\eta^2)\,c\xi+b^2\xi^2\right\}-a^2\,(a^2-\xi^2)(a^2+c\xi)^2$,

and (3) by $b^2\,(b^2-\eta^2)(\xi-c)\,\xi-a^2\,(a^2-\xi^2)\,\eta^2$,

and adding the three products, the coefficients of x and y in the sum will be found to vanish identically, and the absolute term to reduce identically to

$$\left\{\frac{(b^2-\eta^2)(c^2-\xi^2-\eta^2)\,c\xi}{b^2\xi^2+a^2\eta^2}+\xi^2-\eta^2-c^2\right\}(b^6\xi^2+a^6\eta^2-c^4\xi^2\eta^2)\,;$$

but ξ,η satisfy the equation $\frac{a^6}{h^2}+\frac{b^6}{k^2}=c^4$ [*Reprint*, Vol. XXV., p. 43, Solution of Question 4894]; hence the three lines meet in one point.

VII. *Solution by* R. F. DAVIS, B.A.

Employing the figure in Professor Wolstenholme's Solution (September No.), produce SP, S′Q to meet in T. Then O, R′ are obviously centres of escribed circles to the triangle TPS′; and consequently OR′ passes through T and bisects externally the angle at T. Therefore L is the centre of an escribed circle to the triangle TSQ, and LQ bisects the angle SQS′.

4956. (By S. TEBAY, B.A.)—The axis of a parabola is divided into n parts in arithmetical progression by $n-1$ ordinates; find the average area contained between the pth and qth ordinates.

Solution by the PROPOSER.

Let a be the axis, $2b$ the base, and A the area of the parabola. Then

$$y=b\left(\frac{x}{a}\right)^{\frac{1}{2}}, \text{ and } \int y\,dx=\tfrac{1}{2}\text{A}\left(\frac{x}{a}\right)^{\frac{3}{2}}.$$

Let x be the first part; then we have

$$x_p=\tfrac{1}{2}p\,[2x+(p-1)\,d], \qquad x_q=\tfrac{1}{2}q\,[2x+(q-1)\,d]\,;$$

or, since $$a = \tfrac{1}{2}n\,[2x + (n-1)\,d],$$

$$x_p = \frac{p}{n\,(n-1)}\,[a\,(p-1) + n\,(n-p)\,x], \quad x_q = \frac{q}{n\,(n-1)}[a(q-1) + n(n-q)x];$$

and the area between the pth and qth ordinates is

$$\tfrac{2}{3}\mathrm{A}\left\{\left(\frac{x_q}{a}\right)^{\frac{3}{2}} - \left(\frac{x_p}{a}\right)^{\frac{3}{2}}\right\} = \mathrm{X};$$

therefore the average

$$= \frac{1}{a}\int_0^{\frac{a}{n}} \mathrm{X}\,dx = \frac{\mathrm{A}}{5n\,(n-q)}\left\{\frac{p}{n\,(n-1)}\right\}^{\frac{3}{2}}[(n-1)^{\frac{5}{2}} - (q-1)^{\frac{5}{2}}]$$

$$- \frac{\mathrm{A}}{5n\,(n-p)}\left\{\frac{p}{n\,(n-1)}\right\}^{\frac{3}{2}}[(n-1)^{\frac{5}{2}} - (p-1)^{\frac{5}{2}}].$$

Let $n=3$, $p=1$, $q=2$, and the average is $\dfrac{2\,(2\sqrt{2}-1)}{45\sqrt{3}}\,.\,\mathrm{A}.$

4605. (By R. Tucker, M.A.)—Prove that the equation to the circle through the four points in Question 4521 (*Reprint*, Vol. XXIII., pp. 49, 50) is

$$x^2 + y^2 - \left\{h + 8a + 4a\left(\frac{2a-h}{k}\right)^2\right\}x + \left(\frac{k^2 - 4ah + 16a^2}{2k}\right)y$$

$$- \frac{2a}{k^3}\,(2a-h)\,(2h^2 + k^2) = 0,$$

where (h, k) is the point of concurrence of the normals.

Solution by the Proposer.

Let the equation to the circle be $x^2 + y^2 + \mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$, and to the parabola $y^2 = 4ax$; then, at their points of intersection, we have

$$y^4 + 4a\,(\mathrm{A} + 4a)\,y^2 + 16a^2\,\mathrm{B}y + 16a^2\,\mathrm{C} = 0 \quad\text{..................(1).}$$

Referring to my Solution of Question 4521 (*Reprint*, Vol. XXIII., p. 50), if $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ be a point on the parabola, we have

$$m^3k + m^2\,(h-2a) - a = 0, \quad a\mu^3 - \mu\,(h-2a) - k = 0, \text{ if } m\mu = 1;$$

and the ordinates of the points where the normals again cut the parabola, *i.e.*, of the points determined by (1), are given (*loc. cit.*) by $-2a\,(2m+\mu)$, and of the fourth point by $k' = 4a\,\dfrac{2a-h}{k}$, $h' = 4a\left(\dfrac{2a-h}{k}\right)^2$.

Now $$4a\,(\mathrm{A} + 4a) = 4a^2\,[(2m_1 + \mu_1)\,(2m_2 + \mu_2) + \ldots + \ldots] - 2ak'\,(2\Sigma m + \Sigma\mu)$$

$$= 4a^2\,[4\Sigma m_1m_2 + \Sigma\mu_1\mu_2 + 2\,(\Sigma m\,.\,\Sigma\mu - 3)] - 4ak'\,\frac{2a-h}{k}$$

$$= 4a^2\left[\frac{2a-h}{a} - 6\right] - k'^2;$$

therefore $A+4a = -(h+4a)-h'$ and $A = -(h+h'+8a)$,

$$-16a^2B = -8a^3(2m_1+\mu_1)(2m_2+\mu_2)(2m_3+\mu_3)$$
$$+4a^2k'[(2m_1+\mu_1)(2m_2+\mu_2)+\ldots+\ldots];$$

$$\therefore\ 4B = 2a[8m_1m_2\mu_3+4m_1m_2m_3\Sigma\mu^2+2\mu_1\mu_2\mu_3\Sigma m^2+\mu_1\mu_2\mu_3]+k'\left(\frac{h+4a}{a}\right);$$

$$= \frac{2}{k}[k^2-4ah+16a^2],\ \ i.\ e.,\ B = \frac{k^2-4ah+16a^2}{2k};$$

$16a^2C = -k'8a^3(2m_1+\mu_1)(2m_2+\mu_2)(2m_3+\mu_3)$, and $C = -k'\dfrac{2h^2+k^2}{2k}$;

hence the equation required is

$$x^2+y^2-\left[h+8a+4a\left(\frac{2a-h}{k}\right)^2\right]x$$
$$+\frac{k^2-4ab+16a^2}{2k}y-\frac{2a}{k^2}(2a-h)(2h^2+k^2)=0.$$

4648. (By R. TUCKER, M.A.)—Three normals to a parabola intersect in the point (x^2, y^2); prove that the circle drawn through the three corresponding centres of curvature is given by the equation

$$6a(2a-x')(x^2+y^2)-2x(2a-x')(4x'^2-7ax'+10a^2)$$
$$+3ay'(7a-8x')y+4a(8a^3-18a^2x'+15ax'^2-4x'^3)-6ay'^2(a+4x')=0.$$

Solution by the PROPOSER.

The equation to the evolute of the parabola $y^2 = 4ax$ is

$$27ay^2-4(x-2a)^3=0 \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

the equation to its first polar (Salmon's *Higher Plane Curves*, Chap. ii. § iv.) is, after a slight reduction,

$$4(2a-x')(x-2a)^2+9ay(y+2y')=0 \ldots\ldots\ldots\ldots (2);$$

this is the equation to a *conic* passing through the three points of curvature in question.

From (1) and (2) we get

$$27ay^2\,4(2a-n)(x-2a)^2=-4(x-2a)^3\,9ay(y+2y'),$$

that is, $$\frac{y}{2a-x}=\frac{2y'}{4a-3x'+x} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Substituting from (3) in (2), we have, remembering (1),

$$27ay^2+24(x-2a)^2(2a-x')+36(x-2a)(2a-x')^2=108ay'^2 \ldots\ldots (4);$$

$[(4)-3(2)] \div 6$ gives

$$2(x-2a)^2(2a-x')+6(x-2a)(2a-x')^2=18ay'^2+9ayy' \ldots\ldots (5).$$

$6(2)-(4)$ gives

$$27ay^2+108ayy'-36(x-2a)(2a-x')^2+108ay'^2=0 \ldots\ldots\ldots\ldots (6).$$

Multiply (5) by $27a$, and (6) by $2(2a-x')$, and add the results; then we get

$$6a(2a-x')(x^2+y^2)-24a^3(x-a)(2a-x')+18a(x-2a)(2a-x')^2$$
$$-27a^2y'(2y'+y)+24ay'(2a-x')(y+y')-8(x-2a)(2a-x')^3=0.$$

This reduces to

$$6a(2a-x')(x^2+y^2)-2x(2a-x')(4x'^2-7ax'+10a^2)$$
$$+3ay'(7a-8x')y+4a(8a^3-18a^2x'+15ax'^2-4x'^3)-6ay'^2(a+4x')=0,$$

the equation to the circle required.

5017. (By Professor WOLSTENHOLME, M.A.)—If $2a$, $2b$ are the axes of an ellipse or hyperbola, and p the perpendicular from the centre on the tangent at any point, prove that the length of the normal chord at that point is $\dfrac{2a^2b^2}{p\{b^2\pm(a^2-p^2)\}}$.

I. *Solution by* L. WANSBROUGH JONES.

The equation to the curve referred to the tangent and normal, as axes of x and y respectively, may be written

$$Ax^2+2Hxy+By^2-2y=0.$$

If q, p are the coordinates of the centre, we have

$$Aq+Hp=0,\quad Hq+Bp=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Hence, when the equation is transformed to its axes, it takes the form

$$A'x^2+B'y^2=C'=p$$

[since
$$C'=-(Aq^2+2Hqp+Bp^2-2p)$$
$$=-(Aq+Hp)q-(Hq+Bp-1)p+p=p].$$

The length of the normal chord which we get by putting $x=0$ in the original equation is $y=\dfrac{2}{B}$.

Also, solving (1), $\qquad p=\dfrac{A}{AB-H^2}=\dfrac{A}{A'B'}$.

Therefore $\qquad pA'B'+\dfrac{2}{y}=A+B=A'+B'$,

therefore
$$\frac{2}{y}=A'+B'-pA'B'=\frac{p}{a^2}\pm\frac{p}{b^2}\left(1-\frac{p^2}{a^2}\right)$$
$$=\frac{p}{a^2b^2}\left\{b^2\pm(a^2-p^2)\right\},$$

the upper or lower sign being taken according as we have an ellipse or hyperbola; therefore $\qquad y=\dfrac{2a^2b^2}{p\{b^2\pm(a^2-p^2)\}}$.

Hence, in the rectangular hyperbola, the normal chord is always equal and opposite to the diameter of curvature.

II. *Solution by* R. F. Scott, B.A., A. B. Evans, M.A., *and others.*

The normal at x, y has for its equation

$$\frac{\xi - x}{\frac{x}{a^2}} = \frac{\eta - y}{\frac{y}{b^2}} = -\mathrm{R}p,$$

where R is the distance between the points (x, y), (ξ, η). If this be the normal chord,
$$\frac{x^2}{a^2}\left(1-\frac{\mathrm{R}p}{a^2}\right)^2 + \frac{y^2}{b^2}\left(1-\frac{\mathrm{R}p}{b^2}\right)^2 = 1,$$

or
$$2\mathrm{R}p\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right) - \mathrm{R}^2p^2\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}\right) = 0.$$

But $\frac{1}{p^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = \left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right) = \frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{1}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$;

therefore $\frac{x^2}{a^6}+\frac{y^2}{b^6} = \frac{a^2+b^2-p^2}{a^2b^2p^2}$, and therefore $\mathrm{R} = \frac{2a^2b^2}{p\,(a^2+b^2-p^2)}$.

III. *Solution by* Prof. Tanner, M.A., St. John Stephen, *and others.*

Referring the system to two mutually perpendicular tangents as axes, the equations of the conic and the chord of contact are

$$\left(\frac{x}{\alpha}+\frac{y}{\beta}-1\right)^2 - 2\lambda xy = 1,\quad \frac{x}{\alpha}+\frac{y}{\beta} = 1 \;\ldots\ldots\ldots\ldots\; (1, 2).$$

The length of the chord normal to the axes of y is found, by putting $y=\beta$ in (1), to be $n = 2\lambda\alpha^2\beta$. The x-coordinate of the centre, the p of the question, is $p = \frac{\alpha}{2-\lambda\alpha\beta}$. To express these in terms of the semi-axes of the ellipse, we first suppose the conic referred to parallel axes through the centre, $\left(\frac{\alpha}{2-\lambda\alpha\beta}, \frac{\beta}{2-\lambda\alpha\beta}\right)$, and then its equation is

$$\frac{x^2}{\alpha^2}+2\left(\frac{1}{\alpha\beta}-\lambda\right)xy+\frac{y^2}{\beta^2}-\frac{\lambda\alpha\beta}{2-\lambda\alpha\beta} = 0.$$

These axes being rectangular, we have

$$\frac{(2-\lambda\alpha\beta)(\alpha^2+\beta^2)}{\lambda\alpha^3\beta^3} = \frac{1}{a^2}\pm\frac{1}{b^2} = \frac{b^2\pm a^2}{a^2b^2} \text{ and } \frac{(2-\lambda\alpha\beta)^3}{\lambda\alpha^3\beta^3} = \pm\frac{1}{a^2b^2};$$

therefore $\quad a^2 \pm b^2 = \frac{\alpha^2+\beta^2}{(2-\lambda\alpha\beta)^2}$, and $a^2\pm b^2-p^2 = \frac{\beta^2}{(2-\lambda\alpha\beta)^2}$;

therefore $\quad n\times p\times(a^2\pm b^2-p^2) = \frac{2\lambda\alpha^3\beta^3}{(2-\lambda\alpha\beta)^3} = \pm\,2a^2b^2$,

whence we obtain $\quad n = \frac{\pm\,2a^2b^2}{p\,(a^2\pm b^2-p^2)}$

The upper or the lower sign is to be used according as the conic is an ellipse or an hyperbola.

IV. *Solution by the* PROPOSER, R. TUCKER, M.A., *and others.*

The equation of the normal at the point ($a\cos\theta$, $b\sin\theta$) being

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2,$$

we shall have, if ϕ be the excentric angle of the other extremity,

$$\frac{a^2\cos\phi}{\cos\theta}-\frac{b^2\sin\phi}{\sin\theta}=a^2-b^2,$$

or $$\frac{\sin\frac{1}{2}(\phi+\theta)}{b^2\cos\theta}=\frac{\cos\frac{1}{2}(\phi+\theta)}{-a^2\sin\theta}=\frac{\sin\frac{1}{2}(\phi-\theta)}{a^2\sin^2\theta+b^2\cos^2\theta}=\frac{1}{(a^4\sin^2\theta+b^4\cos^2\theta)^{\frac{1}{2}}}$$

$$=\frac{\left\{a^2\sin^2\frac{1}{2}(\phi+\theta)+b^2\cos^2\frac{1}{2}(\phi+\theta)\right\}^{\frac{1}{2}}}{ab\,(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}};$$

whence the length of the normal chord

$$2\sin\tfrac{1}{2}(\phi-\theta)\left\{a^2\sin^2\tfrac{1}{2}(\phi+\theta)+b^2\cos^2\tfrac{1}{2}(\phi+\theta)\right\}^{\frac{1}{2}}$$

is $$\frac{2ab\,(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{a^4\sin^2\theta+b^4\cos^2\theta},\text{ or }\frac{2\left(\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}\right)^{\frac{3}{2}}}{\dfrac{\cos^2\theta}{a^4}+\dfrac{\sin^2\theta}{b^4}};$$

but $\dfrac{1}{p^2}=\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}$, therefore $\dfrac{1}{p^2}\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)=\dfrac{\cos^2\theta}{a^4}+\dfrac{\sin^2\theta}{b^4}+\dfrac{1}{a^2b^2}$,

whence the length $$=2\,\frac{\dfrac{1}{p^3}}{\dfrac{1}{p^2}\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)-\dfrac{1}{a^2b^2}}=\frac{2a^2b^2}{p\,(a^2+b^2-p^2)}.$$

This obviously gives, for the minimum value, $p^2=\frac{1}{3}(a^2+b^2)$, provided this $>b^2$, or if $a^2>2b^2$; and with this value the chord is equal to the radius of curvature, as must generally be the case for a minimum normal chord. The axes will in this case be both maxima values. When $a^2=$ or $<2b^2$, the major axis is a maximum and the minor a minimum (*three* coinciding in it when $a^2=2b^2$), and there are no other maximum or minimum values.

When $a^2+b^2=0$, the normal chord $=-\dfrac{2a^2b^2}{p^3}=-$ diameter of curvature.

This last result was set as a Senate House Question in 1868; it is best proved by the property of the rectangular hyperbola through the corners of a triangle also passing through the centre of perpendiculars; for if P, Q, R be three points on a rectangular hyperbola, S the centre of perpendiculars, and O the centre of the circle P, Q, R; then, when Q, R move up to P, we shall have ultimately SP = 2PO (P being the centroid of the infinitesimal triangle), and O, P, S in one straight line; or PS will be the normal chord at P, and will $=2$PO or the diameter of curvature at P.

[It appears also from the above that the diameter of curvature bears to the normal chord generally the ratio $a^2+b^2-p^2:p^2$.]

5037. (By Sir JAMES COCKLE, F.R.S.)—Given

$$\frac{d^2y}{dx^2}-\frac{1}{\xi}\frac{d\xi}{dx}\frac{dy}{dx}-\xi\eta y=0,\qquad \frac{d^2u}{dx^2}-\frac{1}{\eta}\frac{d\eta}{dx}\frac{du}{dx}-\xi\eta u=0 \;\ldots\ldots\ldots(1,2);$$

find relations between them, and show that the relations suggest a complete symbolical primitive of either of the equations.

Solution by the PROPOSER.

1. Assume $$\frac{dy}{dx}=\xi u \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and (1) becomes $$\frac{d(\xi u)}{dx}-u\frac{d\xi}{dx}-\xi\eta y=0,$$

whence $$\frac{du}{dx}=\eta y \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

2. By (4) we give (2) the form

$$\frac{d(\eta y)}{dx}-y\frac{d\eta}{dx}-\xi\eta u=0,$$

and thus we are led to $$\frac{dy}{dx}=\xi u \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

and I infer that, since (1) and (3) lead to (4), and (2) and (4) to (3), the relations (3) and (4) subsist between particular primitives.

3. From (3) we have $$y=\int\xi u\,dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

whence, by (4), we have $$y=\int\xi\left(\int\eta y\,dx\right)dx,$$

which we may write $$y=\int\xi\,dx\int\eta\,dx\,.\,y \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

hence $$\int\xi\,dx\int\eta\,dx\,.\,y=\int\xi\,dx\int\eta\,dx\int\xi\,dx\int\eta\,dx\,.\,y=y \;\ldots\ldots\ldots(7);$$

and finally, if $\mathrm{P}\int\xi\,dx\int\eta\,dx$ denote the performance of the double integration $\int\xi\,dx\int\eta\,dx$ an infinite number of times, we have

$$y=\mathrm{P}\int\xi\,dx\int\eta\,dx\,.\,y \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

an integration affecting all that follows the integral sign, and the operations being supposed to commence on the right.

4. From (8) we obtain

$$\frac{1}{\xi}\frac{dy}{dx}=\int\eta\,dx\,\mathrm{P}\int\xi\,dx\int\eta\,dx\,.\,y,$$

whence $$\frac{d}{dx}\left(\frac{1}{\xi}\frac{dy}{dx}\right)=\eta y \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

which is equivalent to (1).

5. If arbitrary constants are added after each integration, it is essential

to the identity of the operations that the same constant, say C_2, should be added after the η integration, and, in like manner, C_1 after the ξ integration. For the particular primitives which satisfy (3) and (4) I think that C_1 and C_2 have definite values, probably zero, or (5), (6), and (8) would not be identities. The arbitrary nature of the operations and the transition from (8) to (9) seem to indicate that a particular primitive inserted on the dexter of (8) may give rise to a complete primitive on the sinister.

6. Thus,
$$x\frac{d^2u}{dx^2}+a\frac{du}{dx}-q^2xu=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

discussed by Boole (*Diff. Eq.*, 1865, 2nd ed., pp. 463—474), yields $\xi = x^{-a}$ and $\eta = q^2x^a$; and if we insert the particular primitive $(y=)\,0$, that is, $u=0$, on the dexter of (8), then (8) gives a complete primitive by the successive steps,

$$C_2,$$
$$C_1 + C_2\frac{x^{1-a}}{1-a},$$
$$C_1\frac{q^2x^{a+1}}{a+1} + C_2\left(1+\frac{q^2x^2}{2(1-a)}\right),$$
$$C_1\left(1+\frac{q^2x^2}{2(a+1)}\right) + C_2\left(\frac{x^{1-a}}{1-a}+\frac{q^2x^{3-a}}{2(1-a)(3-a)}\right),$$
$$C_1\left(\frac{q^2x^{a+1}}{a+1}+\frac{q^4x^{a+3}}{2(a+1)(a+3)}\right)+C_2Q,$$

where
$$Q = 1+\frac{q^2x^2}{2(1-a)}+\frac{q^4x^4}{2.4(1-a)(3-a)}.$$

The next steps take us to a result which we may write

$$C_1\left(1+\frac{q^2x^2}{2(a+1)}+\&c.\right)+\frac{C_2x^{1-a}}{1-a}\left(1+\frac{q^2x^2}{2(3-a)}+\&c.\right),$$

and which, taking account of the law of the terms, substantially coincides with that of Boole (p. 464). It is to the even numbered steps that we look in seeking the form of the complete primitive of (10).

7. I notice that, if $\frac{dy}{dx}=\xi y$, then $y = P\int \xi\,dx\,.\,y$, and, inserting 0 for y on the dexter, we get successively

$$C,\quad C\left(1+\int\xi\,dx\right),\quad C\left\{1+\int\xi\,dx+\frac{\left(\int\xi\,dx\right)^2}{1.2}\right\},\quad \text{and so on;}$$

and, ultimately, $y = Ce^{\int\xi\,dx}$, the complete primitive.

8. A desire not to rest the foregoing process merely upon its results fixed my attention upon a passage in De Morgan's "Note," &c. (*Camb. Trans.*, Vol. XI., Part III.), viz., "When $\alpha^\infty y$ is neither 0 nor ∞ it is a finite solution of $\alpha y = y$." De Morgan's α is a functional symbol, but I apprehend that the passage would apply were it an operator, and that it

to some extent supports my assumption. Let $\phi = \int \xi \, dx \int \eta \, dx$, and therefore $\phi^{-1} = \left(\frac{1}{\eta}\frac{d}{dx}\right)\left(\frac{1}{\xi}\frac{d}{dx}\right)$, then I assume that $\mathrm{P}\phi 0$ is a solution, though not necessarily a finite one, of $\phi^{-1}y = y$. The accuracy of the assumption seems demonstrable in certain cases. Thus, let χ be what ϕ becomes when $C_2 = 0$, and let ψ be what ϕ becomes when both C_1 and C_2 vanish. Then $\mathrm{P}\chi 0 = C_1(1 + \psi + \psi^2 + \psi^3 + \&\text{c.})\,1$ ....(11), whereof the dexter satisfies $\psi^{-1}y = y$. But $\mathrm{P}\chi 0 = y$ leads to $\mathrm{P}\chi^{-1}y = 0$, and hence apparently to $y = 0$; that, however, is when, and only when, the ∞ in P transcends the infinite n in ψ^n.

9. Let θ be what ϕ becomes when $C_2 = 0$. Then

$$\mathrm{P}\theta 0 = C_2(1 + \psi + \psi^2 + \&\text{c.})\int \xi \, dx \quad \text{......................(12)},$$

the operands being 0 and $\int \xi \, dx$. Here again the dexter satisfies $\psi^{-1}y = y$; and we also have $\mathrm{P}\theta^{-1}y = 0$ and $y = 0$; but only when in $\theta^{-m}(1 + \psi + \ldots + \psi^n)\int \xi \, dx$ the m and n are infinites related in a certain manner. I conclude that

$$(1 + \psi + \psi^2 + \&\text{c.})\left(C_1 + C_2\int \xi \, dx\right) = y \quad \text{.................. (13)}$$

is a complete synthetical solution of $y = \psi^{-1}y$ or of (9) or of (1), the operations ϕ^{-1}, χ^{-1}, ψ^{-1}, and θ^{-1} being equivalent. When $\mathrm{F} = C_1 + C_2\int \xi \, dx$, then, under the above conditions, $\psi^{-\infty}\,\mathrm{F} = 0$ may be satisfied; but cannot be satisfied when F has any other form. I may add that the synthetical process yields a finite solution of the first "primary form" of Boole.

5056. (By Professor BALL, M.A.)—If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ are four lines in a plane; what is the mechanical meaning of the identical formula $\mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{C}\gamma + \mathrm{D}\delta = 0$, in which A, B, C, D are the areas of the triangles formed by each set of three lines?

Solution by R. W. GENESE, M.A.

The straight line $\mathrm{P}\alpha + \mathrm{Q}\beta + \mathrm{R}\gamma + \mathrm{S}\delta = 0$ is the line of action of the resultant of four forces P, Q, R, S acting along the lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, because it implies that the sum of the moments of the forces about any point in this line is zero. If the four forces be in equilibrium, the equation becomes an identity. To determine the constants in this case, let $\alpha \equiv l_1 x + m_1 y - p_1$, and $l_1^2 + m_1^2 = 1$, $\beta = l_2 x + m_2 y - p_2$, &c. Then

$$\mathrm{P}l_1 + \mathrm{Q}l_2 + \mathrm{R}l_3 + \mathrm{S}l_4 = 0,$$
$$\mathrm{P}m_1 + \mathrm{Q}m_2 + \mathrm{R}m_3 + \mathrm{S}m_4 = 0,$$
$$\mathrm{P}p_1 + \mathrm{Q}p_2 + \mathrm{R}p_3 + \mathrm{S}p_4 = 0;$$

therefore $$\frac{P}{\begin{vmatrix} l_2 & l_3 & l_4 \\ m_2 & m_3 & m_4 \\ p_2 & p_3 & p_4 \end{vmatrix}} = \frac{Q}{\begin{vmatrix} l_3 & l_4 & l_1 \\ m_3 & m_4 & m_1 \\ p_3 & p_4 & p_1 \end{vmatrix}} = \&c.$$

Now $$\Delta(\beta\gamma\delta) = \begin{vmatrix} l_2 & l_3 & l_4 \\ m_2 & m_3 & m_4 \\ p_2 & p_3 & p_4 \end{vmatrix} \div \sin(\beta\gamma)\sin(\gamma\delta)\sin(\delta\alpha),$$

therefore $$\begin{vmatrix} l_2 & l_3 & l_4 \\ m_2 & m_3 & m_4 \\ p_2 & p_3 & p_4 \end{vmatrix} = \left\{\Delta(\beta\gamma\delta)\sin\beta\gamma\sin\gamma\delta\sin\delta\alpha\right\}^{\frac{1}{2}}.$$

Let R_1 be the radius of the circle circumscribing $\beta\gamma\delta$, then

$$\Delta\beta\gamma\delta = 2R_1^2 \sin\beta\gamma\sin\gamma\delta\sin\delta\alpha;$$

therefore $$\begin{vmatrix} l_2 & l_3 & l_4 \\ m_2 & m_3 & m_4 \\ p_2 & p_3 & p_4 \end{vmatrix} = \frac{\Delta\beta\gamma\delta}{2^{\frac{1}{2}}\,.\,R_1}.$$

Thus, if $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ be the areas of the triangles formed by the lines $\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 0$,

$$P:Q:R:S = \frac{\Delta_1}{R_1}:\frac{\Delta_2}{R_2}:\frac{\Delta_3}{R_3}:\frac{\Delta_4}{R_4}.$$

The same result can be readily obtained geometrically by taking moments about the intersections of the lines.

5059. (By the EDITOR.)—If two diagonals of a regular polygon are drawn at random, find (1) the probability that they will intersect within the figure; and (2), if three diagonals are drawn at random, the respective probabilities of 0, 1, 2, 3 interior intersections.

Solution by Professor LLOYD TANNER, M.A.

1. A polygon of n sides has $\frac{1}{2}n(n-3)$ diagonals; for at each of the n corners $(n-3)$ diagonals terminate. There are, consequently, $n(n-3)$ ends of diagonals, and therefore, in all, $\frac{1}{2}n(n-3)$ diagonals.

Suppose one of these diagonals (A) drawn, dividing the given polygon into two polygons of p and $n-p+2$ sides respectively. Then the $\frac{1}{2}n(n-3)$ diagonals of the n-gon are made up of the diagonal A, the $\frac{1}{2}p(p-3)$ diagonals of the p-gon, the $\frac{1}{2}(n-p+2)(n-p-1)$ diagonals of the $(n-p+2)$-gon, and the diagonals which cut A inside the polygon. Thus the number of these last diagonals is

$$\tfrac{1}{2}n(n-3)-1-\tfrac{1}{2}p(p-3)-\tfrac{1}{2}(n-p+2)(n-p-1) = (n-p)(p-2) \ldots(1).$$

Now a second diagonal may be any one of $\frac{1}{2}n(n-3)-1$, that is, any one of $\frac{1}{2}(n^2-3n-2)$ diagonals. Therefore the chance of a second diagonal intersecting A inside the polygon is $\dfrac{2(n-p)(p-2)}{n^2-3n-2}$ .... (2).

But p may have any value from 3 to $n-1$ inclusive, and these values are equally probable. Hence the chance that any two diagonals should intersect inside the polygon is

$$= \frac{1}{n-3} \cdot \frac{2}{n^2-3n-2} \cdot \sum_{p=3}^{p=n-1} (n-p)(p-2)$$

$$= \frac{1}{n-3} \cdot \frac{2}{n^2-3n-2} \cdot \tfrac{1}{6}(n-1)(n-2)(n-3) = \frac{(n-1)(n-2)}{3(n^2-3n-2)} = \text{N say.}$$

2. If there be three diagonals, the chances of 0, 1, 2, 3 internal intersections would be $(1-\text{N})^3$, $3(1-\text{N})^2\,\text{N}$, $3(1-\text{N})\,\text{N}^2$, N^3 respectively.

[In the *Lady's and Gentleman's Diary* for 1857 (p. 55) the above method is applied, by the proposer of this Question, to solve the following problem of Mr. Woolhouse's (*Diary*, Quest. 1904):—

"In a dark room, two persons draw each a chord across a circular slate; what is the chance that they will intersect?"

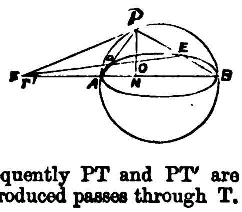
Supposing the polygon regular, and taking the limit of the chance-fraction when n is infinite, we find the required probability to be $\frac{1}{3}$. This result may be otherwise obtained by the Integral Calculus, and it is, in fact, so found in a second solution to Mr. Woolhouse's Question, given in the *Diary* for 1857, by the late Dr. Rutherford.

Professor Tanner's formula (1) may be more simply found by observing that, if each of the $(p-2)$ corners on one side of the diagonal (A) be joined to each of the $(n-p)$ corners on the other side, the $(n-p)(p-2)$ diagonals so obtained will all pass across (A).

If we suppose that the second random diagonal may be any one of the whole series, (A) included, the denominator of the chance-fraction (2) will be n^2-3n, and the final probability will be $\text{N} = \frac{(n-1)(n-2)}{3n(n-3)}$, which agrees with the result given on p. 56 of the above-cited *Diary*.]

4154. (By T. M'Namara.)—From any point P in the circumference of a circle described on the major axis AB of an ellipse a tangent is drawn meeting the axis in T; and PA, PB are drawn intersecting the ellipse in D and E: prove that the chord DE will pass through T.

I. *Solution by the late* T. T. Wilkinson, F.R.A.S.



Let ADEB be the ellipse; APB the circle; and draw the perpendicular PN upon AB, cutting DE in O.

Then, if ED produced does not pass through T, let it cut TA in T'. Then we have

$$\text{P}\,\{\text{T, A, N, E}\} = \text{P}\,\{\text{T}', \text{D, O, E}\},$$

by a known property. But BE, NO, AD are common rays of the homographic system; consequently PT and PT' are common. Hence T' coincides with T, and ED produced passes through T.

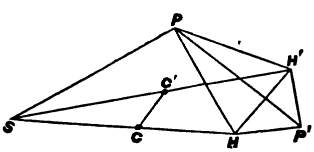
II. *Solution by* Rev. J. L. KITCHIN, M.A.; C. LEUDESDORF, M.A., *and others.*

Taking P to be $(a\cos\theta,\ a\sin\theta)$, the equation to the straight lines PA, PB, is
$$(y-a\sin\theta)^2-(y\cos\theta-x\sin\theta)^2=0,$$
therefore
$$k\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+(y-a\sin\theta)^2-(y\cos\theta-x\sin\theta)^2=0 \text{ ...... (1)},$$
denotes a conic through A, B, D, E. In order that it may represent the straight lines AB, DE, the left-hand side must contain y as a factor. The terms not involving y in (1) are
$$k\left(\frac{x^2}{a^2}-1\right)+a^2\sin^2\theta-x^2\sin^2\theta,$$
which vanish if $k=a^2\sin^2\theta$. When k has this value, (1) becomes
$$y\left\{2x\cos\theta\sin\theta+y\left(1+\frac{a^2}{b^2}\right)\sin^2\theta-2a\sin\theta\right\}=0,$$
shewing that DE cuts $y=0$ at the point $x=a\sec\theta$, which is also the point T where $x\cos\theta+y\sin\theta=a$ cuts AB.

5044. (By J. J. WALKER, M.A.)—If two ellipses have one focus common, and equal major-axes, prove that their real common chord is perpendicular to the line joining their centres.

I. *Solution by* Professor WOLSTENHOLME, M.A.

Let S be the common focus; P, P′ two common points; H, H′ the two second foci; then SP + HP = SP + H′P, or HP = H′P. Similarly HP′ = H′P′; therefore P, P′ are equidistant from H, H′, or PP′ is perpendicular to HH′, and therefore to the line joining the centres, which is parallel to HH′.



II. *Solution by the* PROPOSER.

Let the axes be inclined at an angle α, then the equations to the conics are
$$\frac{p}{r}=1-e\cos\theta,\quad \frac{p'}{r}=1-e'\cos(\theta+\alpha) \text{ .... (1, 2)},$$
whence immediately, for their common chord, we have
$$p'-p=r\left\{(e-e'\cos\alpha)\cos\theta+e'\sin\alpha\sin\theta\right\}.$$

Let $\quad \cot\phi = \dfrac{e' \sin\alpha}{e'\cos\alpha - e}$, then $r\sin(\phi-\theta) + \dfrac{p-p'}{(e^2-2ee'\cos\alpha+e'^2)^{\frac{1}{2}}} = 0$,

so that ϕ is the angle which the common chord makes with the axis of (1). But $\dfrac{e'\sin\alpha}{e-e'\cos\alpha}$ is the tangent of the angle (ψ) which the line joining the centres of the two conics makes with the same axis. Hence $\cot\phi = -\tan\psi$.

4833. (By S. A. Renshaw.)—$acqp$ is a quadrilateral circumscribing a conic; the opposite sides ac, qp meet in r, and ap, cq in b. If the polars of r, q, p meet the directrix in l, m, n respectively, and l, m, n be joined with the focus F, the joining lines meeting pq in e, z, y respectively; prove that be, az, cy meet in a point (O) on the directrix.

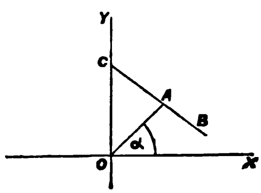
Solution by R. F. Davis, B.A., E. Rutter, *and others.*

This property, like the one exhibited in Question 4674, is the result of a double reciprocation of the theorem that the ortho-centre of a triangle circumscribing a parabola lies on the directrix. Reciprocating once with respect to the focus S, we learn that if S be any point on the circle circumscribing a triangle ABC, and straight lines be drawn through S at right angles to SA, SB, SC, meeting the opposite sides in α, β, γ, these three points lie on a diameter of the circle.* Reciprocating again with respect to any point in the plane of the figure, we get the required result.

[* This follows by Question 4372. Professor Kelland has given a Quaternion proof of this theorem (*Reprint*, Vol. XXIII., p. 30); but a proof by pure geometry will be given on a future page of this volume of the *Reprint*.]

5075. (By T. Cotterill, M.A.)—If a line, of length k, rest upon the curve $(xy-ak)^2 = (x+b)^2(k^2-x^2)$ and the line $x=0$, prove that it envelopes a circle and a parallel to the four-cusped hypocycloid.

Solution by the Rev. F. D. Thomson, M.A.



Let p be the perpendicular from the origin upon the straight line BC, and α the angle which p makes with the axis of x.

Then, if OC $= c$, and (x, y) are coordinates of B, $\quad x = \kappa\sin\alpha$, and $\quad y = c-\kappa\cos\alpha$.

But, from the equation to the curve,

$$(xy-ak)^2 = (x+b)^2(\kappa^2-x^2),$$

therefore $(xy-ak)^2 = (x+b)^2(y-c)^2$, therefore $xy-ak = \pm(x+b)(y-c)$.

First, take upper sign, then $\quad b(y-c)-cx+ak = 0$,

therefore $\qquad -b\cos\alpha - c\sin\alpha + a = 0$, or $p + b\cos\alpha = a$,

or, transferring the origin to $(-b, 0)$, $p = a$, a circle with centre $(-b, 0)$.

Next, taking lower sign, we have

$$2xy + b(y-c) - cx - ak = 0,$$

therefore $\qquad 2\sin\alpha\,(c-\kappa\cos\alpha) - b\cos\alpha - c\sin\alpha - a = 0$,

or $\qquad p - \kappa\sin 2\alpha - b\cos\alpha = a$, or $p = b\cos\alpha + \kappa\sin 2\alpha + a$;

or, transferring the origin to $(b, 0)$, $p = \kappa\sin 2\alpha + a$,

a parallel curve to the hypocycloid $\quad p = \kappa\sin 2\alpha$.

4586. (By Dr. Hart.)—Find n numbers such that, if each be subtracted from the cube of their sum, the remainder shall be cubes.

Solution by the Proposer.

1. For three numbers. Let x, y, z be the numbers, and a = their sum; then $a^3-x=m^3$, $a^3-y=n^3$, $a^3-z=p^3$; whence we have $x=a^3-m^3$, $y=a^3-n^3$, $z=a^3-p^3$, and by addition we have

$$x+y+z = 3a^3-(m^3+n^3+p^3) = a, \quad \text{therefore} \quad m^3+n^3+p^3 = 3a^3-a,$$

where a must be taken $> m, n$, or p, and $< 3a^3$.

Let $a = \dfrac{9}{14}$, then $3a^3-a = \dfrac{423}{(14)^3} = \dfrac{3384}{(28)^3} = \dfrac{1+8+3375}{(28)^3}$.

Take $\quad m = \dfrac{1}{28}, \quad n = \dfrac{2}{28}, \quad p = \dfrac{15}{28}$; then $\quad x = \dfrac{833}{3136}, \quad y = \dfrac{832}{3136}$,

$z = \dfrac{351}{3136}$, or $\dfrac{13}{49}, \dfrac{17}{64}, \dfrac{351}{3136}$, which are the least numbers that have been found.

2. For four numbers. Let w, x, y, z be the numbers, and a = their sum; then $a^3-w=m^3$, $a^3-x=n^3$, $a^3-y=p^3$, $a^3-z=q^3$; whence we have $w=a^3-m^3$, $x=a^3-n^3$, $y=a^3-p^3$, $z=a^3-q^3$; and, by addition,

$$w+x+y+z = 4a^3-(m^3+n^3+p^3+q^3) = a;$$

therefore $m^3+n^3+p^3+q^3 = 4a^3-a$, where a must be taken $> m, n, p$, or q, and $< 4a^3$.

Let $a = \dfrac{5}{9}$, then $4a^3-a = \dfrac{95}{729} = \dfrac{2565}{(27)^3} = \dfrac{27+125+216+2197}{(27)^3}$.

Take $m = \dfrac{3}{27}$, $n = \dfrac{5}{27}$, $p = \dfrac{6}{27}$, $q = \dfrac{13}{27}$; then we have

$$w = \frac{3348}{(27)^3}, \quad x = \frac{3250}{(27)^3}, \quad y = \frac{3159}{(27)^3}, \quad z = \frac{1178}{(27)^3}.$$

3. For five numbers. Let v, w, x, y, z be the numbers, and $a =$ their sum; then $a^3-v = m^3$, $a^3-w = n^3$, $a^3-x = p^3$, $a^3-y = q^3$, $a^3-z = r^3$; whence $v = a^3-m^3$, $w = a^3-n^3$, $x = a^3-p^3$, $y = a^3-q^3$, $z = a^3-r^3$; and, by addition, $v+w+x+y+z = 5a^3-(m^3+n^3+p^3+q^3+r^3) = a$; whence $m^3+n^3+p^3+q^3+r^3 = 5a^3-a$, where a must be taken $> m, n, p, q$, or r, and $< 5a^3$. Let $a = \frac{1}{2}$, then

$$5a^3-a = \frac{1}{8} = \frac{729}{5832} = \frac{1}{5832} + \frac{27}{5832} + \frac{64}{5832} + \frac{125}{5832} + \frac{512}{5832}.$$

Take $m = \frac{1}{18}$, $n = \frac{3}{18}$, $p = \frac{4}{18}$, $q = \frac{5}{18}$, and $r = \frac{8}{18}$; then we have

$$v = \frac{728}{5832},\quad w = \frac{702}{5832},\quad x = \frac{665}{5832},\quad y = \frac{604}{5832},\quad z = \frac{217}{5832}.$$

And in the same way we may proceed to find 6, 7, 8, ... numbers *ad infinitum*. Other classes of problems similar to the above may be solved in ways very much like that employed in the above problem.

When $n=3$, this problem is the same as Problem 19, Book V., of Diophantus's *Arithmetical Questions*, and was considered one of the most difficult of his problems. Bachet, one of the earliest of the modern commentators on Diophantus, solves this problem, as also Hersey and other old authors. On p. 137, Hutton's *Diarian Miscellany*, Wm. Beriffe gives the answer without a solution, and prefixes the following lines:—

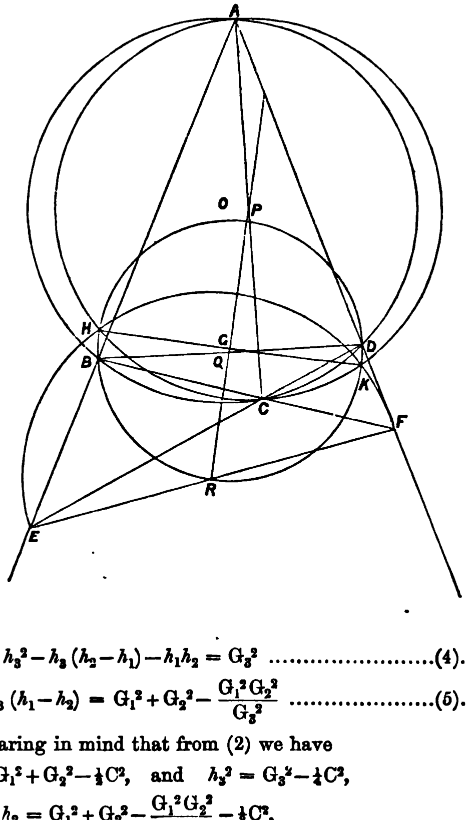
"For those cube numbers three, so nicely to be,
That when every one is subtracted
From the cube of their sum, and yet to remain
Three cubes, has me almost distracted."

On p. 138 Dr. Hutton has given a solution in a note, the answer being $\frac{494424}{2352637}$, $\frac{472696}{2352637}$, $\frac{448000}{2352637}$. These were for a long time thought to be the least numbers. The celebrated Dutch mathematician, Ludolphus van Ceulen, near the close of the 17th century, in his book on the circle, solves this problem and finds the above answers; and so overjoyed was he at his success that he exclaimed, "Constat ergo numeros rite esse inventos. Cujus rei soli Deo debetur gloria." This problem is also solved on p. 52 of Vol. I. of Leybourne's edition of the *Ladies' Diary*. The answers are $\frac{13851}{85184}$, $\frac{18954}{85184}$, $\frac{19467}{85184}$, which were considered as the least. The late Wm. Lenhart, of York, Pennsylvania, has also solved this problem. His solution is to be found in No. V., pp. 266—267, of the *Mathematical Miscellany* for 1837. He solves it by the aid of tables, in which he divides the numbers in the natural series (except those which are cubes) into two cubes. By this means he has found $\frac{4069}{15625}$, $\frac{4032}{15625}$, $\frac{1899}{15625}$ for the numbers, which he believed to be the least; and he thought he had good reason to rejoice over the result, as did Van Ceulen. But the numbers in the annexed solution are much smaller than his, and still less numbers may yet be found.

5058. (By Dr. BOOTH, F.R.S.)—On the diagonals $(2G_1, 2G_2, 2G_3)$ of a complete quadrilateral inscribed in a circle as diameters, circles are drawn; prove that they have a common chord C, whose value is given by the symmetrical expression

$$C^2 = 2(G_1^2+G_2^2+G_3^2) - G_1^2G_2^2G_3^2(G_1^{-4}+G_2^{-4}+G_3^{-4}).$$

Solution by R. TUCKER, M.A.



Let AC $(2G_1)$, BD $(2G_2)$, EF $(2G_3)$ be the three diagonals; P, Q, R their mid-points; then the circles have the common chord HK (a property which is true for *any* complete quadrilateral—McDowell, Ex. 217), and PQR is a straight line, by a well-known property.

Let PQR cut KH in G; then taking $PG = h_1$, $QG = h_2$, $RG = h_3$, we have (McDowell, Ex. 247)

$$PR \,.\, RQ = G_3^2 = (h_3-h_2)(h_3+h_1) \;\;......(1);$$

$$\tfrac{1}{4}C^2 = G_1^2-h_1^2 = G_2^2-h_2^2 = G_3^2-h_3^2 \;...(2);$$

therefore

$$(G_3^2-G_1^2)(G_3^2-G_2^2) = PR\,.\,RQ \times (h_3-h_1)(h_3+h_2) = G_3^2[h_3^2+h_3(h_2-h_1)-h_1h_2] \;...(3).$$

From (1) we have $\quad h_3^2-h_3(h_2-h_1)-h_1h_2 = G_3^2$ ....(4).

From (3) we have $\quad 2h_3(h_1-h_2) = G_1^2+G_2^2-\dfrac{G_1^2G_2^2}{G_3^2}$ ....(5).

Squaring (5), and bearing in mind that from (2) we have

$$h_1^2+h_2^2 = G_1^2+G_2^2-\tfrac{1}{2}C^2, \quad \text{and} \quad h_3^2 = G_3^2-\tfrac{1}{4}C^2,$$

we get
$$2h_1h_2 = G_1^2+G_2^2-\frac{G_1^2G_2^2}{G_3^2}-\tfrac{1}{2}C^2,$$

and
$$4(G_3^2-\tfrac{1}{4}C^2)\frac{G_1^2G_2^2}{G_3^2} = (G_1^2+G_2^2)^2-\frac{2G_1^2G_2^2}{G_3^2}(G_1^2+G_2^2)+\frac{G_1^4G_2^4}{G_3^4}.$$

Hence $\quad C^2 = 2(G_1^2+G_2^2+G_3^2)-G_1^2G_2^2G_3^2(G_1^{-4}+G_2^{-4}+G_3^{-4}).$

[In Dr. Booth's second volume on *New Geometrical Methods*, now passing through the press, the following problem is solved :—

"Given a circle and the three diagonals of the quadrilateral inscribed in it, to construct the quadrilateral."

Let $2G_1$, $2G_2$, $2G_3$ be the three diagonals, as in the above figure; and $PQ = g$; then it is shown that the following equation holds :—

$$\frac{g}{G_3} + \frac{G_2}{G_1} - \frac{G_1}{G_2} = 0, \quad G_1 \text{ being greater than } G_2 \ldots\ldots\ldots\ldots(\alpha).$$

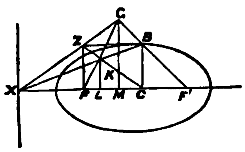
Let C be the common chord HK; then $\frac{1}{2}C$ is the perpendicular from the vertex of the triangle PQH on the line PQ; and if $GQ = x$, the two triangles PHG, QHG give

$$G_1^2 = \tfrac{1}{4}C^2 + (g-x)^2, \quad G_2^2 = \tfrac{1}{4}C^2 + x^2 \ldots\ldots\ldots\ldots\ldots(\beta, \gamma);$$

whence, eliminating x, and substituting the value of g given by (α), we obtain the result stated in the Question.]

4936. (By S. A. Renshaw.)—If F, F′ are the foci of an ellipse, C the centre, X the foot of the directrix belonging to F, and B the vertex of the minor axis: show (1) that F′B is perpendicular to the tangent from X to the auxiliary circle; and (2), if Z be the point of contact of the tangent with the circle, find the relation that must exist between the axes of the ellipse so that, if XB, GF be drawn (G being the point in which FB′ meets the tangent XZ), they may intersect each other on the line ZC.

Solution by H. T. Gerrans, B.A.; L. W. Jones; *and others.*



1. Since the triangles ZCF, BCF′ are equal in all respects; therefore ZC and BF′ are parallel, and XGF′ is a right angle.

2. Let K be the point of intersection of XB, GF, ZC; KL the ordinate; then we have GM : KL = GF : KF = FF′ : CF′,

ZF : GM = ZX : GX = CX : F′X,

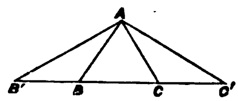
KL : BC = CX : F′X,

therefore KL : GM = $CX^2 : F'X^2$; therefore $CX^2 = 2FX^2$, which gives the relation between the axes.

4935. (By R. Tucker, M.A.)—In a triangle ABC the base BC is produced both ways to B′, C′; prove that the radii of the circles that circumscribe the triangles ABC, ABB′, ACC′, AB′C′ are proportional.

Solution by L. W. JONES; H. T. GERRANS; *and many others.*

Since the triangles ABC, ABB′ have the angles at B supplementary, it is clear that the circumscribed radii are proportional to the sides opposite; that is, as AC : AB′. And since the triangles AC′C, AC′B′ have the angle at C′ common, this is also the ratio of the circumscribed radii of these latter triangles.



INVESTIGATION OF THE VALUE OF $\int_0^\infty \frac{\sin^p x}{x^m}\,dx$, WHEN p, m ARE WHOLE NUMBERS, AND $p > m$. *By* Professor WOLSTENHOLME.

Assuming the well-known results (a, b, ... being positive),

$$\int_0^\infty \frac{\sin ax}{x}\,dx = \tfrac{1}{2}\pi, \quad \int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log\frac{b}{a},$$

and at once deducing from the latter that, if $A + B + C + \ldots = 0$,

$$\int_0^\infty \frac{A\cos ax + B\cos bx + \ldots}{x}\,dx = -A\log a - B\log b - \ldots,$$

we proceed to find the value of $\int_0^\infty \frac{\sin^p x\,dx}{x^m}$, where m, p are whole numbers, and $p > m$.

Let $\quad U = \int_0^\infty \frac{\sin^p x}{x^m}\,dx, \quad$ and $\quad U' = \int_0^\infty \frac{\sin^p ax}{x^m}\,dx,$

where a is positive, we get, by putting $\frac{x}{a}$ for a in the latter,

$$U' = a^{m-1}U, \text{ and therefore } U = \frac{1}{\lfloor m-1}\,\frac{d^{m-1}}{da^{m-1}}U'.$$

CASE I.—When p and m both even, we have

$$2\cos ax = z + \frac{1}{z}, \quad (2i\sin ax)^p = \left(z - \frac{1}{z}\right)^p,$$

or

$$(-1)^{\frac{1}{2}p}.\,2^{p-1}\sin^p ax$$

$$= \cos pax - p.\cos(p-2)ax + \frac{p(p-1)}{\lfloor 2}\cos(p-4)ax - \ldots;$$

therefore $$(-1)^{\frac{1}{2}p}\,2^{p-1}\,U$$

$$= \frac{1}{\lfloor m-1} \frac{d^{m-1}}{da^{m-1}} \int_0^\infty \frac{\cos p\,ax - p\cos(p-2)ax + \frac{p(p-1)}{\lfloor 2}\cos(p-4)\,ax - \ldots}{x^m}$$

$$= \frac{(-1)^{\frac{1}{2}m}}{\lfloor m-1} \int_0^\infty \frac{p^{m-1}\sin pax - p\,(p-2)^{m-1}\sin(p-2)\,ax + \ldots}{x}\,dx$$

$$= \frac{(-1)^{\frac{1}{2}m}}{\lfloor m-1} \cdot \tfrac{1}{2}\pi \left\{p^{m-1} - p\,(p-2)^{m-1} + \frac{p\,(p-1)}{\lfloor 2}(p-4)^{m-1} - \ldots \text{ to } \tfrac{1}{2}p \text{ terms}\right\},$$

or $$U = \frac{\pi}{(-1)^{\frac{1}{2}(p-m)}\lfloor m-1 \,.\, 2^p} \left\{p^{m-1} - p\,(p-2)^{m-1} + \frac{p\,(p-1)}{\lfloor 2}(p-4)^{m-1} - \ldots \text{ to } \tfrac{1}{2}p \text{ terms}\right\}.$$

Case II.—When p and m both odd, we have

$$(-1)^{\frac{1}{2}(p-1)}\,2^{p-1}\sin^p ax =$$

$\sin p\,ax - p\sin(p-2)\,ax + \frac{1}{2}\,[p\,(p-1)]\sin(p-4)ax - \ldots$ to $\frac{1}{2}(p+1)$ terms;

therefore $$(-1)^{\frac{1}{2}(p-1)}\,2^{p-1}\,U$$

$$= \frac{(-1)^{\frac{1}{2}(m-1)}}{\lfloor m-1} \int_0^\infty \frac{p^{m-1}\sin p\,ax - p\,(p-2)^{m-1}\sin(p-2)\,ax + \ldots}{x}\,dx$$

$$= \frac{(-1)^{\frac{1}{2}(m-1)}}{\lfloor m-1} \cdot \tfrac{1}{2}\pi \left\{p^{m-1} - p\,(p-2)^{m-1} + \tfrac{1}{2}p\,(p-1)\,(p-4)^{m-1} - \ldots\ldots \text{ to } \tfrac{1}{2}(p+1) \text{ terms}\right\},$$

or $$U = \frac{\pi}{(-1)^{\frac{1}{2}(p-m)}\lfloor m-1 \,.\, 2^p} \left\{p^{m-1} - p\,(p-2)^{m-1} + \frac{p\,(p-1)}{\lfloor 2}(p-4)^{m-1} - \ldots \text{ to } \tfrac{1}{2}(p+1) \text{ terms}\right\},$$

the same formula as (1) except for the number of terms.

Case III.—When p is odd and m even; then, as in (2),

$$(-1)^{\frac{1}{2}(p-1)}\,2^{p-1}\,U$$

$$= \frac{1}{\lfloor m-1} \frac{d^{m-1}}{da^{m-1}} \int_0^\infty \frac{\sin p\,ax - p\sin(p-2)\,ax + \ldots \text{ to } \frac{1}{2}(p+1) \text{ terms}}{x^m}\,dx$$

$$= \frac{(-1)^{\frac{1}{2}(m-2)}}{\lfloor m-1} \int_0^\infty \frac{p^{m-1}\cos p\,ax - p\,(p-2)^{m-1}\cos(p-2)\,ax + \ldots}{x}\,dx$$

$$= \frac{(-1)^{\frac{1}{2}(m-2)}}{\lfloor m-1} \left\{-p^{m-1}\log p + p\,(p-2)^{m-1}\log(p-2) - \ldots \text{ to } \tfrac{1}{2}(p+1) \text{ terms}\right\},$$

or $$U = \frac{1}{(-1)^{\frac{1}{2}(p-m-1)}\lfloor m-1 \,.\, 2^{p-1}} \left\{p^{m-1}\log p - p\,(p-2)^{m-1}\log(p-2) + \frac{p\,(p-1)}{\lfloor 2}(p-4)^{m-1}\log(p-4) - \ldots \text{ to } \tfrac{1}{2}(p+1) \text{ terms}\right\}.$$

Case IV.—When p is even and m odd; then, as in (1),

$$(-1)^{\frac{1}{2}p} 2^{p-1} \mathrm{U}$$

$$= \frac{1}{\lfloor m-1} \frac{d^{m-1}}{da^{m-1}} \int_0^\infty \frac{\cos pax - p\cos(p-2)ax + \frac{p(p-1)}{\lfloor 2}\cos(p-4)ax - \ldots}{x^m} dx$$

$$= \frac{(-1)^{\frac{1}{2}(m-1)}}{\lfloor m-1} \int_0^\infty \frac{p^{m-1}\cos pax - p(p-2)^{m-1}\cos(p-2)ax + \ldots \text{ to } \tfrac{1}{2}p \text{ terms}}{x} dx$$

$$= \frac{(-1)^{\frac{1}{2}(m-1)}}{\lfloor m-1} \Big\{ -p^{m-1}\log p + p(p-2)^{m-1}\log(p-2)$$
$$- \frac{p(p-1)}{\lfloor 2}(p-4)^{m-1}\log(p-4) + \ldots \text{ to } \tfrac{1}{2}p \text{ terms} \Big\},$$

or $$\mathrm{U} = \frac{1}{(-1)^{\frac{1}{2}(p-m-1)} \lfloor m-1 \,.\, 2^{p-1}} \Big\{ p^{m-1}\log p - \ldots \text{ as in (3), but to } \tfrac{1}{2}p \text{ terms} \Big\}.$$

Of course it has been assumed that $p > m$, otherwise the value of U would be infinite, and the whole may be stated in the two formulæ :

I. If $p-m$ be an even integer,

$$\int_0^\infty \frac{\sin^p x}{x^m} dx = \frac{\pi}{(-1)^{\frac{1}{2}(p-m)} \lfloor m-1 \,.\, 2^p} \Big\{ p^{m-1} - p(p-2)^{m-1} + \frac{p(p-1)}{\lfloor 2}(p-4)^{m-1} - \ldots \Big\},$$

the number of terms being $\frac{1}{2}p$ or $\frac{1}{2}(p+1)$ according as p is even or odd.

II. If $p-m$ be an odd integer,

$$\int_0^\infty \frac{\sin^p x}{x^m} dx = \frac{1}{(-1)^{\frac{1}{2}(p-m-1)} \lfloor m-1 \,.\, 2^{p-1}} \Big\{ p^{m-1}\log p - p(p-2)^{m-1}\log(p-2) + \tfrac{1}{2}p(p-1)(p-4)^{m-1}\log(p-4) - \ldots \Big\},$$

the number of terms being as in I.

It has been assumed in the work that

$$p^{m-1} - p(p-2)^{m-1} + \frac{p(p-1)}{\lfloor 2}(p-4)^{m-1} - \ldots \text{ to } \tfrac{1}{2}p \text{ or } \tfrac{1}{2}(p+1) \text{ terms}$$

is zero when $p-m$ is odd. This is easily proved, but it may be observed that, if it were not so, the above would prove that U must be infinite, whereas it is obviously finite whenever $p > m$, both being positive whether integral or not.

It might easily be proved, by successive integration by parts, that

$$\int_0^\infty \frac{\sin^p x}{x^m} dx = \frac{1}{\lfloor m-1} \int_0^\infty \frac{dx}{x} \frac{d^{m-1}(\sin^p x)}{dx^{m-1}},$$

whence the formulæ could be deduced as above.

[The results are well known, but the foregoing investigation of them is perhaps simpler than any yet given.]

4875. (By R. TUCKER, M.A.)—Find the Radial of a Cassinian oval.

Solution by the PROPOSER.

The equation to a Cassinian oval is

$$\rho^4 - 2\rho^2 c^2 \cos 2\theta + c^4 = m^4 \qquad (1).$$

From Question 4852, the radius of curvature is

$$\rho' = \frac{2m^2\rho^3}{3\rho^4 + c^4 - m^4} \qquad (2).$$

Now $\quad \dfrac{dy}{dx} = -\dfrac{x'}{y'} \cdot \dfrac{\rho^2 - c^2}{\rho^2 + c^2}$; therefore $\cot\theta' = \cot\theta \, \dfrac{\rho^2 - c^2}{\rho^2 + c^2}$ .... (3).

The Radial is obtained by eliminating ρ, θ from (1), (2), (3).

From (3), $\qquad \cos 2\theta = \dfrac{(\rho^4 + c^4)\cos 2\theta' + 2\rho^2 c^2}{\rho^4 + c^4 + 2\rho^2 c^2 \cos 2\theta'}.$

Hence, from (1), $\quad \rho^8 - \rho^4(2c^4 + m^4) - 2m^4c^2 \cos 2\theta' \rho^2 + c^4(c^4 - m^4) = 0,$

and $\qquad 3\rho'\rho^4 - 2m^2\rho^3 + \rho'(c^4 - m^4) = 0.$

Assume $\quad -\lambda = 2c^4 + m^4, \quad \mu = -2m^4c^2\cos 2\theta', \quad \nu = c^4(c^4 - m^4),$

$$-\lambda' = \frac{2m^2}{3\rho'}, \quad \mu' = \frac{c^4 - m^4}{3};$$

then we have to eliminate ρ between

$$\rho^8 + \lambda\rho^4 + \mu\rho^2 + \nu = 0, \quad \rho^4 + \lambda'\rho^3 + \mu' = 0;$$

or $\quad \lambda\mu'\rho^4 + 2\lambda'\mu'^2\rho^3 + (\mu\mu' - \lambda\lambda'^2\mu' + \lambda'^2\nu)\rho^2 + \lambda'^3\nu\rho + \mu'^3 + \mu'\nu - \lambda'^2\mu\mu' = 0,$

$$\rho^4 + \lambda'\rho^3 + \mu' = 0.$$

We may take the equations to be

$$\rho^4 + \kappa\rho^3 + \kappa'\rho^2 + \kappa''\rho + \kappa''' = 0, \quad \rho^4 + \lambda'\rho^3 + \mu' = 0;$$

and the result may be expressed as follows (see SALMON's *Higher Algebra*, p. 34, 1st edition):—

$$\begin{vmatrix} \lambda' - \kappa, & -\kappa', & -\kappa'', & \mu' - \kappa''' \\ -\kappa', & -\kappa'' - \lambda'\kappa', & \mu' - \kappa''' - \lambda'\kappa'', & \kappa\mu' - \lambda'\kappa''' \\ -\kappa'', & \mu' - \kappa''' - \lambda'\kappa'', & \kappa\mu' - \lambda'\kappa''', & \kappa'\mu' \\ \mu' - \kappa''', & \kappa\mu' - \lambda'\kappa''', & \kappa'\mu', & \kappa''\mu' \end{vmatrix} = 0.$$

This, though a fairly simple determinant of the form

$$\begin{vmatrix} p, & a, & b, & c \\ a, & q, & r, & s \\ b, & r, & s, & t \\ c, & s, & t, & u \end{vmatrix} = \begin{aligned} &pqsu + 2prst - pr^2u - ps^3 \\ &\quad - a^2(su - t^2) + b^2(s^2 - qu) - c^2(qs - r^2) \\ &\quad - 2ab(st - ru) - 2ac(rt - s^2) + 2bc(qt - rs) = 0, \end{aligned}$$

is very involved.

Or thus:—Eliminating ρ^4 and the constant terms, and then ρ^3 and the constant terms, from the resulting equations, we get two equations of the form $\qquad ax^2 + bx - c = 0, \quad cx^2 + b'x + c' = 0;$

whence $\qquad (ac' + c^2)^2 = (ab' - bc)(bc' + b'c),$

where
$$
\begin{aligned}
a &= \kappa'(\mu'-\kappa''')-(\kappa-\lambda')(\mu'\kappa-\kappa'''\lambda'),\\
b &= \kappa''(\mu'-\kappa''')-\mu'\kappa'(\kappa-\lambda'),\\
c &= (\mu'-\kappa''')^2+\mu'\kappa''(\kappa-\lambda'),\\
b' &= (\mu'-\kappa''')(\mu'\kappa-\kappa'''\lambda')+\mu'\kappa'\kappa'',\\
c' &= \mu'\kappa'\cdot(\mu'-\kappa''')+\mu'\kappa'^2,\\
\kappa &= 2\mu'.
\end{aligned}
$$

This example furnishes another instance of the extremely complicated forms assumed by the Radials of curves of a higher order than the second.

In *Reprint*, Vol. I., p. 18, the Radial of the Lemniscate ($c=m$) is found to be $r^2 \cos\frac{2}{3}\theta = \frac{1}{9}c^2$.

4973. (By Prof. Townsend, F.R.S.)—Defining, as an axis of momental symmetry of a rigid body, every axis for all planes passing through which the body has a common moment of inertia, and as axes of equimomental symmetry, all such axes for which the common moments of inertia are equal; show that—

(a) Through every point of a rigid body pass two axes of equimomental symmetry; which axes are the two generators of the skew hyperboloid through the point confocal with the central ellipsoid of gyration of the body.

(b) Every system of axes of equimomental symmetry in a rigid body consists of two groups; which groups are the two opposite systems of generators of the same skew hyperboloid confocal with the aforesaid ellipsoid of the body.

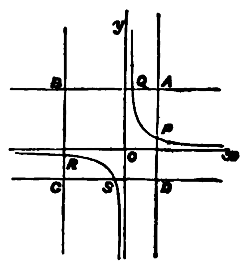
Solution by the Proposer.

Both these properties are immediate consequences from the two well known properties—(a) that, for a rigid body, every system of equimomental planes envelopes a quadric of the system confocal with the central ellipsoid of gyration of the body; and, conversely, every system of planes enveloping a quadric of the aforesaid confocal system is an equimomental system of the body; and (b) that, for a skew quadric or any other scroll, every plane passing through any generator is a tangent plane to the surface.

5030. (By C. Leudesdorf, M.A.)—Given a fixed straight line AB, and a point P, lying within the circle on AB as diameter. In AB two points X, Y are taken at random; find (1) the chance that the circle on XY as diameter includes P; and (2) show that this chance becomes an even one when AB is infinite.

I. *Solution by* E. B. Elliott, M.A.

1. Let the length of the perpendicular PN from P on AB be c, and the parts AN, NB into which it divides AB be a and b respectively. Then, by supposition, $ab > c^2$. Now let x be the distance of X from N measured towards A, and y that of Y from N measured towards B. Then in favourable cases



$$x > -b < a, \quad y > -a < b,$$

and $$xy > c^2.$$

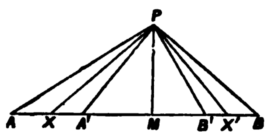
Take now rectangular axes and construct the equilateral hyperbola $xy = c^2$, and the straight lines $x = a$, $x = -b$, $y = -a$, $y = b$. Then the chance required is

$$= \frac{\text{APQA} + \text{CRSC}}{\text{ABCD}} = \frac{2\int_{\frac{c^2}{b}}^{a}\int_{\frac{c^2}{x}}^{b} dx\,dy}{(a+b)^2} = \frac{2}{(a+b)^2}\left\{ab - c^2 - c^2\log\frac{ab}{c^2}\right\}.$$

2. This chance clearly becomes $\frac{1}{2}$ when $a = b = \infty$.

II. *Solution by the* Proposer.

1. Let $\text{AB} = 2a$, $\text{AM} = h$, $\text{MP} = k$. Join PA, PB, and draw PB′, PA′ at right angles to them. Then we have



$$\text{AA}' = \frac{2ah - (h^2+k^2)}{2a-h},$$

$$\text{BB}' = \frac{2a - (h^2+k^2)}{h}.$$

Clearly, neither X nor Y can lie in A′B′. If X lie in AA′, let $\text{AX} = x$; join PX and draw PX′ perpendicular to it. Then Y must lie within X′B;

and because $\text{AX}' = h + \dfrac{k^2}{h-x}$, the chance of this is $\dfrac{2a - h - \dfrac{k^2}{h-x}}{2a}$; thus the chance required is, if X lie in AA′,

$$\frac{1}{\text{AB}}\int_0^{\text{AA}'}\left(2a - h - \frac{k^2}{h-x}\right)\frac{dx}{\text{AA}'}$$

$$= \frac{1}{\text{AB}\,.\,\text{AA}'}\left[\text{MB}\,.\,\text{AA}' + \text{PM}^2\log\frac{\text{A}'\text{M}}{\text{AM}}\right] \;\ldots\ldots\; (1).$$

So, if X lie in BB′, the chance is

$$= \frac{1}{\text{AB}\,.\,\text{BB}'}\left[\text{AM}\,.\,\text{BB}' + \text{PM}^2\log\frac{\text{B}'\text{M}}{\text{BM}}\right] \;\ldots\ldots\ldots\; (2);$$

therefore total chance required

$$= \frac{AA'}{AB}(1) + \frac{BB'}{AB}(2)$$

$$= \frac{1}{AB^2}\left[MB \cdot AA' + AM \cdot BB' + PM^2 \log \frac{A'M \cdot B'M}{AM \cdot BM}\right] \;\ldots\ldots\; (3),$$

$$= \frac{1}{2a^2}\left[a^2 - (h^2 + k^2) + k^2 \log \frac{k^2}{a^2 - h^2}\right] \;\ldots\ldots\ldots\ldots\; (4),$$

the origin being tranferred to the middle point of AB.

2. When $a = \infty$, (4) becomes $\frac{1}{2}$; also when $h^2 + k^2 = a^2$ it vanishes; and when $k = 0$, it becomes $\frac{a^2 - h^2}{2a^2}$; all just as it should do.

If the cotangents of PAB, PBA be denoted by α, β respectively, the expression (3) may be written in the form $2 \cdot \frac{\alpha\beta - 1 - \log \alpha\beta}{(\alpha + \beta)^2}$.

III. *Solution by the* Rev. H. G. Day, M.A.

Let the distance XY be d, and $AB = a$; then the chance of the random point P being included in the circle on XY as diameter is evidently the average value of $\frac{d^2}{a^2} = \frac{1}{6}$, since it is obvious that the average value of d^n is $\frac{2}{(n+1)(n+2)} a^n$. This ratio is unaltered when a becomes infinite.

5077. (By E. Lemoine.)—Placer un triangle O′A′B′, donné en grandeur, de façon que O′ coincide avec le sommet O d'un triangle OAB, donné en grandeur et en position, et que les droites AA′, BB′ fassent entre elles un angle donné. Examen particulier du cas où l'angle est nul.

Solution by R. W. Genese, M.A.

Let us suppose the triangle placed as required. Turn the triangle BOB′ about O till OB′ falls on OA′; that is, through the known angle B′OA′. Let CC′ be the new positions of BB′; then CC′ makes a known angle with AA′ (viz., the sum or difference of the given angle and B′O′A′). Through A′ draw A′D parallel to C′C, meeting OC in D. Then D is a known point, for OD : OB = OA′ : OB′, and angle BOD = angle B′OA′. The point A′ can therefore be determined by the intersection of a circle, of centre O and radius OA′, with a segment of a circle on AD containing the known angle DA′A.

4138. (By S. WATSON.)—Two points A, B are taken at random upon the surface of a given circle, centre O; find the chance that the circle through O, A, B shall be less than one-fourth of the given circle.

Solution by C. LEUDESDORF, M.A.

Taking O as origin, let A be (r, θ), B $(r', \theta+\phi)$, and c = radius.

Then, area of circle round OAB $= \pi\left(\dfrac{\text{AB}}{2\sin\phi}\right)^2 = \dfrac{\pi}{4}\,\dfrac{r^2+r'^2-2rr'\cos\phi}{\sin^2\phi}$;

and if this be less than $\frac{1}{4}\pi c^2$, we must have

$$c^2\cos^2\phi - 2rr'\cos\phi + r^2 + r'^2 - c^2 < 0;$$

therefore $\cos\phi$ must lie between $\dfrac{rr' \pm \left\{(c^2-r^2)(c^2-r'^2)\right\}^{\frac{1}{2}}}{c^2}$,

and the chance of this is $\dfrac{2\left\{(c^2-r^2)(c^2-r'^2)\right\}^{\frac{1}{2}}}{2c^2}$;

therefore the chance required $= \displaystyle\int_0^\pi \frac{d\theta}{\pi}\int_0^c\int_0^c \frac{\left\{(c^2-r^2)(c^2-r'^2)\right\}^{\frac{1}{2}}}{c^2}\,\frac{dr}{c}\,\frac{dr'}{c}$

$$= \frac{1}{c^4}\left\{\int_0^c (c^2-r^2)^{\frac{1}{2}}\,dr\right\}^2 = \frac{1}{c^4}\left(\frac{\pi c^2}{4}\right)^2 = \frac{\pi^2}{16}.$$

5079. (By Professor CAYLEY, F.R.S.)—Show that the curve

$$\left\{(\beta-\gamma i)^2-\delta^2\right\}^{\frac{1}{2}}\left\{(x-\beta i)^2+y^2\right\}^{\frac{1}{2}} + q\left\{(\beta+\gamma i)^2-\delta^2\right\}^{\frac{1}{2}}\left\{(x+\beta i)^2+y^2\right\}^{\frac{1}{2}}$$
$$= \left\{(1-q^2)\frac{\beta}{\delta}\right\}^{\frac{1}{2}}\left\{\beta^2-(\gamma-\delta i)^2\right\}^{\frac{1}{2}}\left\{(x-\gamma-\delta i)^2+y^2\right\}^{\frac{1}{2}}$$

[where $i = (\sqrt{-1})$ as usual] is a real bicircular quartic having the axial foci βi, $-\beta i$, $\gamma+\delta i$, $\gamma-\delta i$.

Solution by the PROPOSER.

Consider the equation

$$(l+mi)^{\frac{1}{2}}\left[(x-\beta i)^2+y^2\right]^{\frac{1}{2}} + q\,(l-mi)^{\frac{1}{2}}\left[(x+\beta i)^2+y^2\right]^{\frac{1}{2}}$$
$$= (\lambda+\mu i)^{\frac{1}{2}}\left[(x-\gamma-\delta i)^2+y^2\right]^{\frac{1}{2}}.$$

This is

$$(l+mi)\left\{x^2+y^2-\beta^2-2\beta xi\right\} + q^2(l-mi)\left\{x^2+y^2-\beta^2+2\beta xi\right\}$$
$$-(\lambda+\mu i)\left\{x^2+y^2-\beta^2+\beta^2+\gamma^2-\delta^2-2\gamma x-2(x-\gamma)\delta i\right\}$$
$$+2q\,(l^2+m^2)^{\frac{1}{2}}\left[(x^2+y^2-\beta^2)^2+4\beta^2x^2\right]^{\frac{1}{2}} = 0,$$

where, putting the imaginary part equal zero, we have

$$m(1-q^2)(x^2+y^2-\beta^2) - 2l(1-q^2)\beta x$$
$$- \mu\left\{x^2+y^2-\beta^2+(\beta^2+\gamma^2-\delta^2)-2\gamma x\right\} + 2\lambda(x-\gamma)\delta = 0,$$

which will be true identically if

$$m(1-q^2)-\mu=0, \quad -l(1-q^2)\beta+\mu\gamma+\lambda\delta=0, \quad -\mu(\beta^2+\gamma^2-\delta^2)-2\lambda\gamma\delta=0.$$

The last gives $\lambda=\theta(\beta^2+\gamma^2-\delta^2)$, $\mu=-2\theta\gamma\delta$, θ arbitrary;

and then $l(1-q^2)\beta = \theta\delta(\beta^2+\gamma^2-\delta^2-2\gamma^2) = \theta\delta(\beta^2-\gamma^2-\delta^2)$,

$$m(1-q^2) = -2\theta\delta\gamma;$$

so that, putting $\theta\delta=(1-q^2)\beta$, or $\theta=(1-q^2)\dfrac{\beta}{\delta}$, we have

$$l=\beta^2-\gamma^2-\delta^2, \quad m=-2\beta\gamma,$$
$$\lambda=(1-q^2)\frac{\beta}{\delta}(\beta^2+\gamma^2-\delta^2), \quad \mu=(1-q^2)\frac{\beta}{\delta}(-2\gamma\delta);$$

therefore $l\pm mi=(\beta\mp\gamma i)^2-\delta^2$, $\lambda\pm\mu i=(1-q^2)\dfrac{\beta}{\delta}[\beta^2+(\gamma\mp\delta i)^2]$,

and the equation is

$$[(\beta-\gamma i)^2-\delta^2]^{\frac{1}{2}}[(x-\beta i)^2+y^2]^{\frac{1}{2}} + q[(\beta+\gamma i)^2-\delta^2]^{\frac{1}{2}}[(x+\beta i)^2+y^2]^{\frac{1}{2}}$$
$$= \left((1-q^2)\frac{\beta}{\delta}\right)^{\frac{1}{2}}[\beta^2-(\gamma-\delta i)^2]^{\frac{1}{2}}[(x-\gamma-\delta i)^2+y^2]^{\frac{1}{2}},$$

which is a real curvo having the axial foci $+\beta i$, $-\beta i$; $\gamma+\delta i$; $\gamma-\delta i$; viz., $\gamma+\delta i$ being a focus, and the curve being real, it is clear that $\gamma-\delta i$ is also a focus.

5039. (By Prof. Townsend, F.R.S.)—The orbit of a comet, supposed parabolic, being supposed to intersect that of a planet, supposed elliptic, at a pair of diametrically opposite points; determine for the latter, supposed given, the diameters of longest and shortest interterminal description by the comet.

Solution by the Proposer.

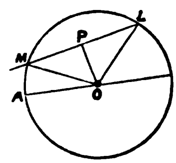
Denoting by r_1 and r_2 the distances of the extremities of any diameter $2r$ of the given planetary orbit from the sun; then, since by Lambert's Theorem (see Hymers's *Elements of Astronomy*, 2nd edit., Art. 325), the time t of inter-terminal description by the comet for that diameter

$$= \frac{1}{6(\mu)^{\frac{1}{2}}}[(r_1+r_2+2r)^{\frac{3}{2}}+(r_1+r_2-2r)^{\frac{3}{2}}] = \frac{1}{3}\left(\frac{2}{\mu}\right)[(a+r)^{\frac{3}{2}}+(a-r)^{\frac{3}{2}}],$$

where μ = the mass of the sun and a = the mean distance of the orbit, the diameters for which t is greatest and least are consequently those for which r is greatest and least, that is, the two axes of the orbit, and therefore, &c.

5063. (By B. WILLIAMSON, M.A.)—A is a fixed point on the circumference of a circle. Two other points L, M are taken on the circumference such that arc AL = m arc AM, where m is any constant; prove (1) that the envelope of the straight line LM is an epicycloid or a hypocycloid, according as the arcs AL and AM are measured in the same or in opposite directions, on the circumference; and (2) show how to construct the fixed circle, and also the rolling circle for any position.

Solution by the Rev. F. D. THOMSON, M.A.



Draw OP from the centre perpendicular to ML. Let OP $= p$, AOM $= \theta$, AOL $= m\theta$. Then POA $= \frac{1}{2}(1+m)\,\theta$, and the equation to ML, referred to the axes through O, is

$$x\cos\tfrac{1}{2}(1+m)\,\theta + y\sin\tfrac{1}{2}(1+m)\,\theta = p,$$

which is satisfied by $x = a\cos\theta$, $y = a\sin\theta$; therefore

$$a\left[\cos\theta\cos\tfrac{1}{2}(1+m)\,\theta + \sin\tfrac{1}{2}(1+m)\,\theta\sin\theta\right] = p,$$

or
$$p = a\cos\tfrac{1}{2}(m-1)\,\theta.$$

Let $\frac{1}{2}(1+m)\,\theta = \alpha$, then $p = a\cos\frac{m-1}{m+1}\alpha$, which is the tangential equation to an epicycloid formed by circles whose radii are $\frac{a}{m+1}$, $\frac{m-1}{m+1}a$.

The point R, in which two consecutive tangents intersect, is given by

$$\text{LR} : \text{MR} = \text{arc LL}' : \text{arc MM}' = m : 1, \quad \text{therefore} \quad \text{LR} = \frac{m}{m+1}\,\text{ML}.$$

The perpendicular to ML through R will meet the fixed circle, of centre O and radius $\frac{m-1}{m+1}a$, in the point where the rolling circle touches it. Hence the centre of the latter is found.

Writing $-m$ for m, the equation becomes $p = a\cos\frac{m+1}{m-1}\theta$, which denotes a hypocycloid; and the radii of the circles are $\frac{a}{m-1}$, $\frac{m+1}{m-1}a$.

5082. (By Prof. CROFTON, F.R.S.)—1. If θ be the vertical angle of a triangle whose base is a fixed line AB ($=2c$), and (x, y) be the vertex, prove that $\iint \sin\theta\, dx\, dy = 8c\,(a-c)$, the integration extending over an ellipse whose foci are A, B, and whose axis is $2a$.

2. Show that the above integral remains unchanged in value, if the integration extend over any Cartesian oval whose *internal* foci are A, B, and whose axis is $2a$. [All such Cartesians pass through the extremities

of the axis minor of the ellipse. An instance of one is a circle from A as centre, with a as radius; provided $a > 2c$.]

Solution by E. B. Elliott, M.A.

If r, r' be the distances of the vertex (x, y) from A, B, we have

$$rr' \sin\theta = 2\Delta = 2cy.$$

Thus,
$$\iint \sin\theta \, dx\, dy = \iint \frac{2cy}{rr'} dx\, dy \quad \text{.........................(1)}.$$

Now $r^2 = (x+c)^2 + y^2$, $r'^2 = (x-c)^2 + y^2$; so that

$$\frac{dr}{dx} = \frac{x+c}{r}, \quad \frac{dr}{dy} = \frac{y}{r}, \quad \frac{dr'}{dx} = \frac{x-c}{r'}, \quad \frac{dr'}{dy} = \frac{y}{r'}.$$

Therefore the Jacobian
$$\begin{vmatrix} \dfrac{dr}{dx}, & \dfrac{dr}{dy} \\ \dfrac{dr'}{dx}, & \dfrac{dr'}{dy} \end{vmatrix} = \frac{2cy}{rr'}.$$

Thus between any corresponding limits

$$\iint \frac{2cy}{rr'} dx\, dy = \iint dr\, dr' \quad \text{...........................(2)}.$$

1. To find the value of this integral when the integration extends over the elliptic area $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$ or $r + r' = 2a$, let us first find its value over the infinitely small space between the two confocals $r + r' = 2a$, $r + r' = 2(a + da)$. It is clearly

$$= 2\int_{a-c}^{a+c} \int_{2a-r}^{2(a+da)-r} dr\, dr' = 4da \int_{a-c}^{a+c} dr = 8c\, da.$$

Now integrate this for a between lower limit c and upper limit a, and we obtain at once that over the whole area of the ellipse

$$\iint \sin\theta \, dx\, dy = 8c\,(a-c).$$

2. More generally, to find the value over the interior of the Cartesian oval $mr + (1-m)\,r' = a$ (m being less than unity), we have first, for its value over the infinitesimal space between the two confocals corresponding to a and $a + da$,

$$2\int_{a-2c(1-m)}^{a+2c(1-m)} \int_{\frac{a-mr}{1-m}}^{\frac{a+da-mr}{1-m}} dr\, dr' = \frac{2da}{1-m} \int_{a-2c(1-m)}^{a-2c(1-m)} dr = 8c\, da;$$

and integrating this between lower limit c and upper limit a, we see that over the whole area of the Cartesian the value of the integral is, as before, $8c\,(a-c)$.

[Prof. Crofton's solution is given on p. 192 of a very interesting paper on *Local Probability*, published in the *Philosophical Transactions* (of the Royal Society) for 1868.]

5047. (By C. LEUDESDORF, M.A.)—Given a plane lamina bounded by a closed curve; let A_0, A be the areas of the pedals formed from the given curve by taking (1) the centre of gravity (G) of the lamina, (2) any point P on the lamina, as pedal origin. Also let k be the radius of gyration of the lamina about an axis through G perpendicular to its plane. Show that the average value of A (P ranging all over the lamina) is $A_0 + \frac{1}{2}\pi k^2$.

I. *Solution by* R. F. SCOTT, B.A.

Let p, p_1 be the perpendiculars from P and G on any tangent; ψ the angle the tangent makes with one of the principal axes through G; a, b the radii of gyration about these principal axes; Δ = area of lamina.

$$A = \tfrac{1}{2}\int p^2\, d\psi;$$

$$2\Delta\,.\,\text{mean value of A} = \iint p^2\, d\psi\,.\,d\Delta = \Delta\ (a^2\cos^2\psi + b^2\sin^2\psi + p_1{}^2)\, d\psi$$

$$= \pi\Delta\,(a^2+b^2) + 2A_0\,\Delta.$$

Therefore the mean value of $A = \frac{1}{2}\pi k^2 + A_0$.

[See also the Solution of Question 1435, *Reprint*, Vol. I., pp. 35—36.]

II. *Solution by* E. B. ELLIOTT, M.A.

Let p be the perpendicular on any tangent to the bounding curve from P, and p_0 that from G. Also let ϕ be the angle made by these perpendiculars with GP, and r be the length of GP. Then we have

$$p = p_0 - r\cos\phi;$$

therefore

$$A = \tfrac{1}{2}\int_0^{2\pi} p^2 d\phi = \tfrac{1}{2}\int_0^{2\pi} p_0{}^2 d\phi + \tfrac{1}{2}r^2\int_0^{2\pi}\cos^2\phi\, d\phi - p_0 r\int_0^{2\pi}\cos\phi\, d\phi = A_0 + \tfrac{1}{2}\pi r^2.$$

Therefore, also, the average value of $A = A_0 + \frac{1}{2}\pi k^2$.

III. *Solution by the* PROPOSER.

Let G be taken as origin of coordinates, and let P be (x, y); let also p, p' be the perpendiculars from G and P on a tangent to the closed curve, making an angle ϕ with the initial line. Then we have

$$A = \tfrac{1}{2}\int_0^{2\pi} p'^2 d\phi = \tfrac{1}{2}\int_0^{2\pi}(p - x\cos\phi - y\sin\phi)^2 d\phi$$

$$= A_0 + \tfrac{1}{2}(x^2+y^2)\pi - \int_0^{2\pi} p\,(x\cos\phi + y\sin\phi)\, d\phi;$$

therefore the average value of A

$$= \iint A\, dx\, dy \div \Omega \quad (\text{if } \Omega = \text{area of lamina})$$

$$= A_0 + \frac{\pi}{2\Omega}\iint (x^2+y^2)\, dx\, dy - \iiint_0^{2\pi} p\,(x\cos\phi + y\sin\phi)\, d\phi\, dx\, dy.$$

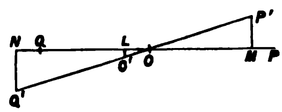
But, since the origin is the centre of gravity,

$$\iint x\,dx\,dy = \iint y\,dx\,dy = 0\,; \quad \text{also} \quad \iint (x^2+y^2)\,dx\,dy = \Omega k^2\,;$$

therefore the average value of $A = A_0 + \frac{1}{2}\pi k^2$.

5076. (By the Rev. A. F. Torry, M.A.)—A chord PQ cuts off a constant area from a given oval curve; show that the radius of curvature of its envelop will be $\frac{1}{4}$PQ $(\cot\theta + \cot\phi)$, θ and ϕ being the angles at which PQ cuts the given curve.

I. *Solution by* R. F. Davis, B.A.; R. W. Genese, M.A.; *and others.*

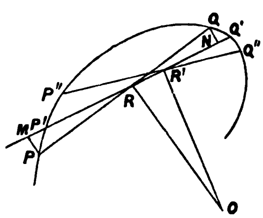


By a known theorem PQ touches the envelop at its middle point O. Let P′OQ′ be the consecutive chord, and O′ its middle point. Then, drawing the perpendiculars P′M, O′L, Q′N upon PQ, it follows, since the triangles POP′, QOQ′ are equal, that $P'M = Q'N = \beta$. Also

$$OL = \tfrac{1}{2}(PM + QN) = \tfrac{1}{2}\beta(\cot\theta + \cot\phi), \text{ and the angle } POP' = \frac{2\beta}{PQ}.$$

$$\text{Hence radius of curvature} = \frac{OL}{POP'} = \tfrac{1}{4}PQ\,(\cot\theta + \cot\phi).$$

II. *Solution by the* Rev. F. D. Thomson, M.A.



Let PQ, P′Q′, P″Q″ be consecutive chords. Then, ultimately,

area PRP′ = QRQ′;

therefore PR = RQ; similarly, P′R′ = R′Q′.

Draw PM, QN perpendicular to P′, Q′. Then, ultimately, PR = MR;

therefore $\quad P'R = PR - MP'$;

therefore
$$\begin{aligned} RR' &= P'R' - P'R \\ &= P'R' - PR + MP' \\ &= \tfrac{1}{2}(P'Q' - PQ) + MP'. \end{aligned}$$

But $\quad P'Q' = QR + Q'M + PR - MP' = PQ + Q'M - MP'$;

therefore $\;P'Q' - PQ = Q'N - P'M$; therefore $RR' = \frac{1}{2}(Q'N + P'M)$.

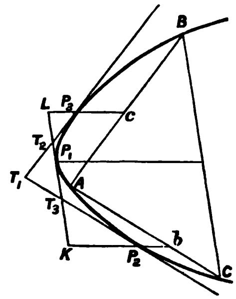
Let $\delta\psi$ be the angle between PQ and P′Q′; then

$$\begin{aligned} \rho\delta\psi = RR' = \tfrac{1}{2}(Q'N + P'M) &= \tfrac{1}{2}PM\,(\cot\phi + \cot\theta) \\ &= \tfrac{1}{4}PQ\,\delta\psi\,(\cot\phi + \cot\theta); \end{aligned}$$

therefore $\qquad \rho = \frac{1}{4}PQ\,(\cot\phi + \cot\theta)$.

4854. (By S. A. RENSHAW.)—If ABC be a triangle inscribed in a parabola, and $T_1T_2T_3$ the circumscribed triangle that has its sides parallel to those of ABC, prove that the sides of ABC are four times those of $T_1T_2T_3$.

Solution by R. F. DAVIS, B.A.



Let P_1, P_2, P_3 be the points of contact of the sides of the triangle $T_1T_2T_3$; and b, c the middle points of CA, AB. Produce the diameters P_2b, P_3c to meet T_2T_3 in K, L. Then $bKLc$ is a parallelogram, and we have

$$KL = bc = \tfrac{1}{2}BC.$$

Also, since the diameter through T_2 bisects P_3P_1, therefore $T_2P_1 = T_2L$.

Similarly, $\quad T_3P_1 = T_3K$.

Hence $\quad T_2T_3 = \tfrac{1}{2}KL = \tfrac{1}{4}BC.$

[Other Solutions are given on p. 86 of Vol. XXV. of the *Reprint*.]

5080. (By Prof. SYLVESTER, F.R.S.)—If the coordinates (x, y, z) of a curve be proportional to three given cubic functions of t; prove the existence and find the positions of the node, in terms of the coefficients, and extend the method to the case of unicursal curves of any order.

Solution by Professor WOLSTENHOLME, M.A.

Suppose the point (x, y, z) to be given in terms of a parameter θ by

$$x : y : z = a_1\theta^3 + b_1\theta^2 + c_1\theta + d_1 : a_2\theta^3 + b_2\theta^2 + c_2\theta + d_2 : a_3\theta^3 + b_3\theta^2 + c_3\theta + d_3.$$

Then, for a node, we must have the same point for two different values of the parameter, θ, ϕ suppose, and we have three equations of the type

$$a_1(\theta^3-\lambda\phi^3) + b_1(\theta^2-\lambda\phi^2) + c_1(\theta-\lambda\phi) + d_1(1-\lambda) = 0,$$

whence $$\frac{\theta^3-\lambda\phi^3}{(bcd)} = \frac{\theta^2-\lambda\phi^2}{(cda)} = \frac{\theta-\lambda\phi}{(dab)} = \frac{1-\lambda}{(abc)},$$

or $\theta^3-\lambda\phi^3$, $\theta^2-\lambda\phi^2$, $\theta-\lambda\phi$, $1-\lambda$ are as the four determinants

$$\left\|\begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{matrix}\right\|$$

which denote by A, B, C, D. We have then

$$\theta^3-\lambda\phi^3 = \mu A, \quad \theta^2-\lambda\phi^2 = \mu B, \quad \theta-\lambda\phi = \mu C, \quad 1-\lambda = \mu D,$$

whence
$$\mu^2(B^2-AC) = \lambda\theta\phi(\theta-\phi)^2,$$
$$\mu^2(C^2-BD) = \lambda(\theta-\phi)^2,$$
$$\mu^2(BC-AD) = \lambda(\theta+\phi)(\theta-\phi)^2,$$

[illegible]

[illegible] $\theta + \phi =$ [illegible] $\theta\phi =$ [illegible]

[illegible] when [illegible] = 0 [illegible]

[illegible]

and [illegible] whence $\theta + \phi = 0$, $\theta\phi = -\frac{1}{3}$,

and these imaginary values of θ, ϕ give the real node $x = -by = z$; but it is an acnode, the tangents being

$$[illegible] = 0, \quad \text{or} \quad by = ax - x^2 [illegible]$$

[illegible] obvious [illegible]

Another method of determining the node is as follows:—

If θ be the parameter, and $\theta_1, \theta_2, \theta_3$ the three values of θ for collinear points, there must be one relation between $\theta_1, \theta_2, \theta_3$, since two points determine the straight line. If we make this relation an identity as to θ_3, the necessary conditions will determine θ_1, θ_2, the two values of θ, either of which gives the node. Thus to take the same example as before, if $px + qy + rz = 0$ be a straight line, the points where it meets the curve [illegible] are given by

$$p\,\theta(\theta - 1)^2 + q\theta^2 + r(\theta + 1)^2 = 0;$$

therefore $\quad \theta_1 + \theta_2 + \theta_3 = \dfrac{[illegible]}{p + q + r}, \quad \theta_1\theta_2\theta_3 = \dfrac{[illegible]}{p + q + r},$

so that $\quad \theta_1 + \theta_2 + \theta_3 = 3\theta_1\theta_2\theta_3.$

The node is then determined by $\theta_1 + \theta_2 = 0$, and $3\theta_1\theta_2 = 1$, the same results of course, as before. It is obvious that this method may easily be extended to curves of higher orders. I do not know if this method has been previously given; it was suggested to me some years ago, either by a remark of a pupil or by a question in a Downing College examination paper.

For a curve of higher order, as the fourth, if θ be the parameter, and $\theta_1, \theta_2, \theta_3, \theta_4$ the values of θ for four collinear points, there will be two relations between $\theta_1, \theta_2, \theta_3, \theta_4$, or one relation between $\theta_1, \theta_2, \theta_3$ (this will always be so); and if we make this relation an identity in θ_3, the necessary conditions will give equations for θ_1, θ_2, the two values of θ, at a node; and if these values be equal, the node will be a cusp.

Another exemplification of this method of determining the node of a cubic is the curve $\quad x : y : z = \cos 3\theta : \sin 3\theta : \cos\theta.$

If θ, ϕ, ψ be three collinear points,

$$\begin{vmatrix} \cos 3\theta, & \cos 3\phi, & \cos 3\psi \\ \sin 3\theta, & \sin 3\phi, & \sin 3\psi \\ \cos\theta, & \cos\phi, & \cos\psi \end{vmatrix} = 0,$$

or, rejecting the factors $\sin(\phi - \psi)$, &c.,

$$\cos(\phi + \psi - \theta) + \cos(\psi + \theta - \phi) + \cos(\theta + \phi - \psi) = 0,$$

or $$\cos\psi\,[\cos(\phi + \theta) + 2\cos(\phi - \theta)] + \sin\psi\,\sin(\theta + \phi) = 0.$$

Hence, for a node,

$$\cos(\phi+\theta)+2\cos(\phi-\theta)=0,\quad \sin(\theta+\phi)=0;$$

$$\theta+\phi=0,\quad \cos(\phi-\theta)=-\tfrac{1}{2},\quad \text{or}\quad \theta=\tfrac{1}{3}\pi=-\phi.$$

This gives the node $(2:0:-1)$; and, obtaining the equation of the curve

$$(x^2+y^2)(x+3z)=4z^3,$$

we find that at this point there are the two real tangents

$$(x+2z)\sqrt{3}=\pm y.$$

If we put $z=a$, we get the circular cubic $y^2=x^2\dfrac{3a-x}{a+x}$; and if $z=a-x-y$, the curve whose polar equation is

$$r=\frac{a}{\cos\theta+\sin\theta+\cos\frac{1}{3}\theta},$$

which has the node, $(-2a,\ \pi)$, $(2a,\ 2\pi)$. The curve is readily traced from the equations $\dfrac{x}{\cos\theta}=\dfrac{y}{\sin\theta}=\dfrac{a-x-y}{\cos\frac{1}{3}\theta}$,

and has three real asymptotes.

On applying the method to the curve

$$x:y:z=\theta^4+1:\theta^3+\theta:2\theta^2,$$

there is failure; but in this case, if we put $\theta+\dfrac{1}{\theta}=\phi$, we have

$$x:y:z=\phi^2-2:\phi:2,$$

so that the curve is a conic $[2y^2=z(x+z)]$ and has no node.

Another example is

$$x:y:z=\cos 2\theta:\sin 2\theta:\cos\theta;$$

whence, if θ, ϕ, ψ be three collinear points,

$$\begin{vmatrix} \cos 2\theta, & \cos 2\phi, & \cos 2\psi \\ \sin 2\theta, & \sin 2\phi, & \sin 2\psi \\ \cos\theta, & \cos\phi, & \cos\psi \end{vmatrix}=0,$$

or $\cos\theta+\cos\phi+\cos\psi+\cos(\phi+\psi-\theta)+\cos(\psi+\theta-\phi)+\cos(\theta+\phi-\psi)=0$,

which is an identity in ψ, if

$$1+\cos(\theta+\phi)+2\cos(\theta-\phi)=0,\quad \sin(\theta+\phi)=0,\quad \cos\theta+\cos\phi=0,$$

all of which are satisfied if $\theta+\phi=\pm\pi$, $\theta-\phi=\pm\frac{1}{2}\pi$; giving the two nodes $(0,\ \pm\sqrt{2},\ 1)$.

If $z=a$, this is the curve whose polar equation is $r\cos\frac{1}{2}\theta=a$, and it is obvious from the figure that there is a third node at infinity. This corresponds to $\theta=\frac{1}{2}\pi$, $\phi=-\frac{1}{2}\pi$, or is in general the points $(1,\ 0,\ 0)$.

The necessary and sufficient conditions for a node are

$$\cos\tfrac{1}{2}(\theta-\phi)=0,\quad \cos(\theta+\phi)=1,$$

and these give the three nodes belonging to a quartic of deficiency 0.

It is obvious that, in a unicursal curve of the nth degree, if a, b, c be

the values of the parameter at three collinear points, the relation between a, b, c (when the factors $b-c$, $c-a$, $a-b$ have been suppressed) will be of the degree $n-2$ in a; so that, to make this an identity in a, we shall have $n-1$ relations between b, c. Since this must happen at a node, these relations must be equivalent to two only, which leads to singular algebraical theorems, of which perhaps the *à priori* proof might be difficult. As an example of this happening, take the curve

$$x : y : z = a^4 : 1+a^2 : a.$$

For three collinear points a, b, c

$$\begin{vmatrix} a^4, & b^4, & c^4 \\ 1+a^2, & 1+b^2, & 1+c^2 \\ a, & b, & c \end{vmatrix} = 0,$$

or $$abc\,(a+b+c) = a^2+b^2+c^2+bc+ca+ab,$$

that is, $$a^2(bc-1)+a\,(b+c)\,(bc-1)-(b^2+bc+c^2) = 0,$$

giving, for an identity in a, the two relations

$$bc = 1, \quad b^2+bc+c^2 = 0,$$

or there are the two acnodes

$$(-1,\ 1,\ 1)\,;\quad (1,\ -1,\ 1).$$

There is of course a third node, corresponding to the values $+\infty - a$ of the parameter, at the points (1, 0, 0), and the failure of the method to indicate this appears to be due to the incompleteness of the functions for y and z; the missing terms, I should expect, needing to be supplied with zero coefficients.

4787. (By H. W. Harris.)—Prove that

$$3\left\{\frac{\sin^2(\theta-\alpha)}{\sin^2(\alpha-\beta)\sin^2(\alpha-\gamma)}+\frac{\sin^2(\theta-\beta)}{\sin^2(\beta-\alpha)\sin^2(\beta-\gamma)}+\frac{\sin^2(\theta-\gamma)}{\sin^2(\gamma-\alpha)\sin^2(\gamma-\beta)}\right\}$$

$$\times\left\{\frac{\sin^5(\theta-\alpha)}{\sin^5(\alpha-\beta)\sin^5(\alpha-\gamma)}+\frac{\sin^5(\theta-\beta)}{\sin^5(\beta-\gamma)\sin^5(\beta-\alpha)}+\frac{\sin^5(\theta-\gamma)}{\sin^5(\gamma-\alpha)\sin^5(\gamma-\beta)}\right\}$$

$$=5\left\{\frac{\sin^3(\theta-\alpha)}{\sin^3(\alpha-\beta)\sin^3(\alpha-\gamma)}+\frac{\sin^3(\theta-\beta)}{\sin^3(\beta-\alpha)\sin^3(\beta-\gamma)}+\frac{\sin^3(\theta-\gamma)}{\sin^3(\gamma-\alpha)\sin^3(\gamma-\beta)}\right\}$$

$$\times\left\{\frac{\sin^4(\theta-\alpha)}{\sin^4(\alpha-\beta)\sin^4(\alpha-\gamma)}+\frac{\sin^4(\theta-\beta)}{\sin^4(\beta-\alpha)\sin^4(\beta-\gamma)}+\frac{\sin^4(\theta-\gamma)}{\sin^4(\gamma-\alpha)\sin^4(\gamma-\beta)}\right\}$$

$$=\frac{30}{7}\left\{\frac{\sin^7(\theta-\alpha)}{\sin^7(\alpha-\beta)\sin^7(\alpha-\gamma)}+\frac{\sin^7(\theta-\beta)}{\sin^7(\beta-\alpha)\sin^7(\beta-\gamma)}+\frac{\sin^7(\theta-\gamma)}{\sin^7(\gamma-\alpha)\sin^7(\gamma-\beta)}\right\}.$$

Solution by the PROPOSER.

Let $\lambda_1, \lambda_2, \lambda_3$ denote the quantities

$$\frac{\sin(\theta-\alpha)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)},\quad \frac{\sin(\theta-\beta)}{\sin(\beta-\alpha)\sin(\beta-\gamma)},\quad \frac{\sin(\theta-\gamma)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)};$$

then $\lambda_1+\lambda_2+\lambda_3 = 0$. Hence $\lambda_1, \lambda_2, \lambda_3$ may be regarded as roots of an equation of the form $x^3+qx+r=0$. Now, since every symmetric function of the roots of this equation of the 7th degree must be of the form Kq^2r (where K is a numerical factor), it may easily be proved that

$$3(\lambda_1^2+\lambda_2^2+\lambda_3^2)(\lambda_1^5+\lambda_2^5+\lambda_3^5) = 5(\lambda_1^3+\lambda_2^3+\lambda_3^3)(\lambda_1^4+\lambda_2^4+\lambda_3^4)$$
$$= \tfrac{30}{7}(\lambda_1^7+\lambda_2^7+\lambda_3^7).$$

[Other Solutions are given on pp. 88, 89 of Vol. XXIV. of the *Reprint*.]

5050. (By A. MARTIN.)—In Barlow's *Theory of Numbers*, page 299, it is stated that "the equation $x^2-5658y^2=1$ has for its least values

$x = 16610072525797731839820799846220132470201461350$3,

$y = 698253616416770487157775940222021002391003072$."

Show (1) that these numbers are not correct, and (2) that the true values are $x = 1284836351$, $y = 17081120$.

Solution by the PROPOSER.

1. The units figure of the square of Barlow's value of x is 9; the units figure of the square of his value of y is 4; the units figure of $5658y^2$ is 2, and $9-2=7$. If his values were correct, the units figure of x^2 wou'd be 1 greater than the units figure of $5658y^2$, unless the units figure of x^2 was 0, in which case the units figure of $5658y^2$ would be 9.

2. Let $A=5658$; then

$$\sqrt{(5658)} = \sqrt{A} = r+\cfrac{1}{u_1+\cfrac{1}{u_2+\cfrac{1}{u_3+\ \&c.,}}}$$

where r is the greatest integer contained in $\sqrt{A}$. The last quotient in every complete period is $2r$. Let m be the number of quotients in a complete period, and $\frac{p_m}{q_m}$ the last convergent in the first period; then, when m is even, $x=p_m$, $y=q_m$.

Let $\quad \frac{\sqrt{A}+a_n}{b_n} = u_n + \&c.,\quad$ and $\quad \frac{\sqrt{A}+a_{n+1}}{b_{n+1}} = u_{n+1}+\&c.$

be any two consecutive complete quotients; then

$$a_{n+1} = u_nb_n - a_n,\quad b_{n+1} = \frac{A-a_{n+1}^2}{b_n}.$$

If $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$ be any two consecutive convergents, and u_{n+1} the quotient corresponding to $\frac{p_{n+1}}{q_{n+1}}$; then

$$\frac{p_{n+2}}{q_{n+2}} = \frac{u_{n+1}\,p_{n+1} + p_n}{u_{n+1}\,q_{n+1} + q_n}.$$

$$\begin{array}{llll}
\frac{\sqrt{(5658)}+0}{1} = 75 + = r, & & \frac{r+}{33} = 4 + = u_1, \\
\frac{r+57}{73} = 1 + = u_2, & & \frac{r+16}{74} = 1 + = u_3, \\
\frac{r+58}{31} = 4 + = u_4, & & \frac{r+66}{42} = 3 + = u_5, \\
\frac{r+60}{49} = 2 + = u_6, & & \frac{r+38}{86} = 1 + = u_7, \\
\frac{r+48}{39} = 3 + = u_8, & & \frac{r+69}{23} = 6 + = u_9, \\
\frac{r+69}{39} = 3 + = u_{10}, & & \frac{r+48}{86} = 1 + = u_{11}, \\
\frac{r+38}{49} = 2 + = u_{12}, & & \frac{r+60}{42} = 3 + = u_{13}, \\
\frac{r+66}{31} = 4 + = u_{14}, & & \frac{r+58}{74} = 1 + = u_{15}, \\
\frac{r+16}{73} = 1 + = u_{16}, & & \frac{r+57}{33} = 4 + = u_{17}, \\
\frac{r+75}{1} = 150 + = u_{18} = 2r. & & &
\end{array}$$

As 18, the number of quotients in a complete period of $\sqrt{(5658)}$, is even, therefore

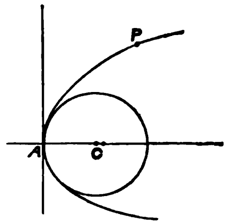
$$x = p_{18} = rq_{18} + q_{17} = 1284836351,$$
$$y = q_{18} = 17081120.$$

5066. (By R. F. Davis, B.A.)—Circles are drawn touching a fixed straight line in such a manner that one of their number intersects orthogonally all the others. Show that the locus of the centres is a parabola.

I. *Solution by* Prof. Wolstenholme.

The circle $x^2 + y^2 + 2gx + 2fy - R^2 = 0$ cuts orthogonally $x^2 + y^2 - R^2 = 0$, and will touch $y = R$ if $x^2 + 2gx + 2fR + 2R^2 = 0$ has equal roots; that is, if $g^2 = 2R(f + R)$, so that the centre lies on the parabola $x^2 = 2R(y + R)$. The enunciation of the question is somewhat incorrect, the circle which

cuts the system at right angles cannot be one of the system, or it must cut itself at right angles, which is only the case with a point-circle. I presume all that the proposer meant by it was, that the single circle also touches the fixed straight line. If C be the centre of the circle of curvature at the vertex A of a parabola, and any circle be drawn with its centre (P) on the parabola and touching the tangent at the vertex, this circle will cut the circle of curvature at the vertex orthogonally. For with the ordinary notation



$$CP^2 = y^2 + (x-2a)^2 = 4ax + (x-2a)^2 = x^2 + 4a^2$$
$$= \text{sum of squares on the radii};$$

or the circles cut orthogonally.

II. *Solution by* L. W. JONES, M.A., *the* PROPOSER, *and others.*

If from any point P (x, y) on the parabola $y^2 = 4ax$, tangents be drawn to the osculating circle at the vertex, the (length)^2 of either of these tangents $= (x-2a)^2 + y^2 - 4a^2 = x^2$. Hence the circle described with P as centre, and with the perpendiculars from P on the tangent at the vertex as radius, intersects orthogonally the osculating circle at the vertex.

5081. (By Prof. TOWNSEND, F.R.S.)—A straight line being supposed to intersect at angles α and β an ellipse whose semi-axes are a and b; required the length of the chord intercepted by it on the curve.

I. *Solution by* Professor WOLSTENHOLME, M.A.

If θ, ϕ be the excentric angles of the ends of the chord, we have

$$\tan\alpha = \frac{\frac{a}{b}\left\{\tan\theta - \tan\frac{1}{2}(\theta+\phi)\right\}}{1+\frac{a^2}{b^2}\tan\theta\tan\frac{1}{2}(\theta+\phi)} = \frac{ab\sin\frac{1}{2}(\theta-\phi)}{a^2\sin\theta\sin\frac{1}{2}(\theta+\phi)+b^2\cos\theta\cos\frac{1}{2}(\theta+\phi)}$$

$$= \frac{2ab\sin\frac{1}{2}(\theta-\phi)}{(a^2+b^2)\cos\frac{1}{2}(\theta-\phi) - (a^2-b^2)\cos\left[\theta+\frac{1}{2}(\theta+\phi)\right]},$$

whence

$$\cot\alpha = \left(\frac{a^2+b^2}{2ab} - \frac{a^2-b^2}{2ab}\cos(\theta+\phi)\right)\cot\tfrac{1}{2}(\theta-\phi) + \frac{a^2-b^2}{2ab}\sin(\theta+\phi),$$

and similarly

$$\cot\beta = \left(\frac{a^2+b^2}{2ab} - \frac{a^2-b^2}{2ab}\cos(\theta+\phi)\right)\cot\tfrac{1}{2}(\theta-\phi) - \frac{a^2-b^2}{2ab}\sin(\theta+\phi),$$

and $\quad \cot\alpha + \cot\beta = \dfrac{2}{ab}\left[a^2\sin^2\tfrac{1}{2}(\theta+\phi) + b^2\cos^2\tfrac{1}{2}(\theta+\phi)\right]\cot\tfrac{1}{2}(\theta-\phi),$

$$\cot\alpha - \cot\beta = \frac{a^2-b^2}{ab}\sin(\theta+\phi).$$

Also, if l be the length of the chord, we have

$$l^2 = a^2(\cos\theta-\cos\phi)^2 + b^2(\sin\theta-\sin\phi)^2$$
$$= 4\sin^2\tfrac{1}{2}(\theta-\phi)\,[a^2\sin^2\tfrac{1}{2}(\theta+\phi) + b^2\cos^2\tfrac{1}{2}(\theta+\phi)],$$

or
$$\frac{l^2}{2\sin^2\frac{1}{2}(\theta-\phi)} = a^2+b^2-(a^2-b^2)\cos(\theta+\phi)$$
$$= a^2+b^2-[(a^2-b^2)^2-a^2b^2(\cot\alpha-\cot\beta)^2]^{\frac{1}{2}}.$$

Again, $\quad a^2b^2(\cot\alpha+\cot\beta)^2 = \dfrac{l^4}{4\sin^4\frac{1}{2}(\theta-\phi)}\cot^2\tfrac{1}{2}(\theta-\phi)$

$$= \left\{a^2+b^2-[(a^2-b^2)^2-a^2b^2(\cot\alpha-\cot\beta)^2]^{\frac{1}{2}}\right\}^2\left\{\frac{2}{l^2}[a^2+b^2-(\ldots\ldots)^{\frac{1}{2}}-1]\right\},$$

whence $\quad \dfrac{2}{l^2}\left\{a^2+b^2-[(a^2-b^2)^2-a^2b^2(\cot\alpha-\cot\beta)^2]^{\frac{1}{2}}\right\}$

$$= a^2b^2(\cot\alpha+\cot\beta)^2+\left\{a^2+b^2-[(a^2-b^2)^2-a^2b^2(\cot\alpha-\cot\beta)^2]^{\frac{1}{2}}\right\}^2$$
$$= 2\left\{a^4+b^4+2a^2b^2\cot\alpha\cot\beta-(a^2+b^2)[(a^2-b^2)^2-a^2b^2(\cot\alpha-\cot\beta)^2]^{\frac{1}{2}}\right\},$$

or, finally,

$$l^2 = \frac{\left\{a^2+b^2-[(a^2-b^2)^2-a^2b^2(\cot\alpha-\cot\beta)^2]^{\frac{1}{2}}\right\}^3}{a^4+b^4+2a^2b^2\cot\alpha\cot\beta-(a^2+b^2)[(a^2-b^2)^2-a^2b^2(\cot\alpha-\cot\beta)^2]^{\frac{1}{2}}}.$$

Of course the radical may be taken with either sign, but must be taken the same in numerator and denominator.

If the chord be a tangent, $l=0$; but that does not happen by the vanishing of the numerator of the above, but by the denominator becoming infinite ($\cot\alpha\cot\beta$).

II. *Solution by the* Rev. F. D. Thomson, M.A.

Let the equation to the conic be

$$ax^2+by^2+2hxy-1 = 0 \qquad\ldots\ldots(1).$$

The equation to the pair of tangents whose chord of contact is $y-c=0$ is

$$ax^2+by^2+2hxy-1 = \kappa(y-c)^2, \text{ where } \kappa = \frac{ab-h^2}{a-c^2(ab-h^2)} \ldots (2, 3).$$

But when $y=c$, we have $\quad ax^2+2hxc+bc^2-1=0$, from (1);

therefore $\quad (x-x')^2 = \dfrac{4h^2c^2-4a(bc^2-1)}{a^2} = \dfrac{4}{a^2}[a-c^2(ab-h^2)]$:

therefore, if the length of the chord be $2l$, $l^2 = \dfrac{a-c^2(ab-h^2)}{a^2}$;

therefore, from (3), we have $\quad \kappa = \dfrac{ab-h^2}{a^2l^2} \qquad\ldots\ldots(4).$

Again, if m be the tangent of the angle which one of the lines (2) makes with axis of y, we have

$$am^2+2hm+b-\kappa = 0;\ \text{therefore } m_1+m_2 = -\frac{2h}{a},\quad m_1m_2 = \frac{b-\kappa}{a}.$$

Also, if A, B be the reciprocals of the squares of the semiaxes,

$$a+b = A+B,\quad ab-h^2 = AB;$$

therefore $\kappa = \dfrac{AB}{a^2l^2}$, also $b = A+B-a = am_1m_2+\kappa = am_1m_2+\dfrac{AB}{a^2l^2}$;

therefore $$a^3(1+m_1m_2)-a^2(A+B)+\frac{AB}{l^2}=0 \quad\ldots\ldots\ldots\ldots(\alpha);$$

also $$a^2+h^2=a(A+B)-AB=a^2+\tfrac{1}{4}a^2(m_1+m_2)^2;$$

or $$a^2[4+(m_1+m_2)^2]-4a(A+B)+4AB=0\ldots\ldots\ldots\ldots(\beta).$$

Hence l is given by (α) in terms of a, which is a root of (β).

Cor.—The corresponding condition for a parabola may be obtained by putting $\frac{A}{B^2}=p^2$, $a=rB$, and then putting $B=0$; we thus get

$$4l(m_1-m_2)=p[4+(m_1+m_2)^2]^{\frac{3}{2}}.$$

4729. (By the Editor.)—Two tangents to a semi-cubical parabola include a given angle; find (1) the locus of their point of intersection, and (2) what the locus becomes when the given angle is a right angle.

I. *Solution by* Stephen Watson.

Let (x', y') be the point of contact of one of the tangents, and m the tangent of its inclination to the axis of x; then we have

$$3ay'^2=2x'^3,\quad m=\frac{x'^2}{ay'},\quad \text{therefore}\quad x'=\tfrac{3}{2}am^2,\quad y'=\tfrac{3}{2}am^3;$$

hence the equation of the tangent, $y-y'=m(x-x')$, becomes

$$2am^3-9xm+9y=0\quad\ldots\ldots\ldots\ldots(1).$$

Let u, u', p be the values of m in (1); then we have

$$u+u'+p=0,\quad uu'+p(u+u')=-\frac{9x}{2a},\quad uu'p=-\frac{9y}{2a};$$

therefore $$(u+u')^2-uu'=\frac{9x}{2a},\quad uu'(u+u')=\frac{9y}{2a}\ldots\ldots\ldots\ldots(2,3).$$

But if t be the tangent of the given angle, we have

$$u-u'=t(1+uu')\quad\ldots\ldots\ldots\ldots(4).$$

From (2) and (4) we easily obtain the values of uu' and $u+u'$, and these values substituted in (3) give the equation of the locus required.

When the given angle is a right angle, $uu'=-1$, and therefore, by (2) and (3), $$\frac{9x}{2a}-1=\frac{81y^2}{4a^2},\quad\text{or}\quad 81y^2=2a(9x-2a),$$

which is the equation of a parabola, and needs no explanation.

II. *Solution by the* Proposer.

Let $x^3=ay^2$ be the equation of the curve, and (x_1, y_1), (x_2, y_2) the points of contact of two tangents which contain a given angle a $(=\tan e)$; then the equation of the required locus will be found by eliminating y_1, y_2 from

$$a^{\frac{1}{3}}(2y+y_1) = 3xy_1^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots (1),$$

$$a^{\frac{1}{3}}(2y+y_2) = 3xy_2^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots (2),$$

$$a^{\frac{1}{3}}(y_1^{\frac{1}{3}}-y_2^{\frac{1}{3}}) = c\,(\tfrac{2}{3}y_1^{\frac{1}{3}}y_2^{\frac{1}{3}}+\tfrac{4}{3}a^{\frac{2}{3}}) \quad \ldots\ldots\ldots\ldots (3).$$

where (1), (2) are the equations of the tangents, and (3) is the condition that they contain an angle $\alpha\,(=\tan^{-1}c)$.

From $\{(1)-(2)\}$ we have $a^{\frac{1}{3}}(y_1^{\frac{2}{3}}+y_1^{\frac{1}{3}}y_2^{\frac{1}{3}}+y_2^{\frac{2}{3}}) = 3x$ .... (4).

From $\{y_1^{\frac{1}{3}}(1)-y_2^{\frac{1}{3}}(2)\}$ we have $y_1^{\frac{1}{3}}y_2^{\frac{1}{3}}(y_1^{\frac{1}{3}}+y_2^{\frac{1}{3}}) = 2y$ .... (5).

From $\{a^{\frac{1}{3}}(4)-(3)\}$ we have

$$a^{\frac{1}{3}}y_1^{\frac{1}{3}}y_2^{\frac{1}{3}} = (\tfrac{4}{3}ac^{-2}x+\tfrac{14}{27}a^2c^{-2}+\tfrac{4}{9}a^2c^{-4})^{\frac{1}{2}}-\tfrac{2}{3}ac^{-2}-\tfrac{4}{3}a = z \text{ suppose} \ldots\ldots (6).$$

From $\{(4)+(6)\}$ we have $a^{\frac{1}{3}}(y_1^{\frac{1}{3}}+y_2^{\frac{1}{3}})^2 = z+3x$ .... (7).

From (5), (6), (7), we find the equation of the required locus to be

$$(3x+z)\,z^2 = 4ay^2 \quad \ldots\ldots\ldots\ldots (8).$$

If $\alpha = \frac{1}{2}\pi$, then $c = \infty$; and, taking $4x^3 = 27my^2$ as the equation of the curve, where $4m$ is the parameter of the parabolic involute, the equation (8) becomes $y^2 = m\,(x-m)$. The locus is then, in this case, a parabola whose parameter is one-fourth of that of the involute of the given curve.

Since Mr. WATSON's a is equal to $\frac{4}{3}m$, it is at once apparent that his result agrees with this.

III. *Solution by* R. TUCKER, M.A.

The tangents to the semicubical parabola are the normals to a parabola, of which let the equation be $y^2 = 4ax$; then the locus of the point of section (x, y) is given by eliminating m between the equations

$$am^3+(2a-x)\,m+y = 0 \quad \ldots\ldots\ldots\ldots (1),$$

$$\lambda m^3(2a-x-\lambda y)+m^2\,[\lambda^2(2a-x)+2x-a+3\lambda y]$$
$$+\,m\,[\lambda(2x-a)-3y]+a\lambda^2+2a-x = 0 \quad \ldots\ldots\ldots\ldots (2),$$

where (2) is derived from (1) by substituting $\dfrac{m+\lambda}{1-\lambda m}$ for m, λ being the tangent of the given constant angle.

If the given angle be a right angle, $\lambda = \infty$, and the equations become

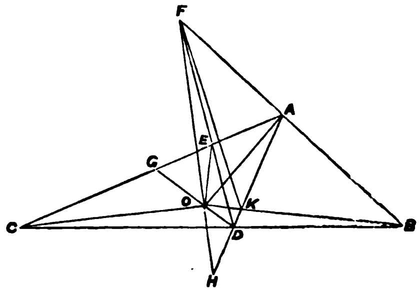
$$am^3+(2a-x)\,m+y = 0, \quad ym^3-(2a-x)\,m^2-a = 0.$$

The elimination in this case is readily effected, and gives the well-known result (easily got otherwise)

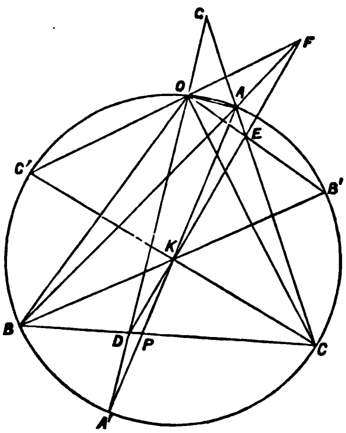
$$y^2 = a\,(x-3a).$$

4372. (By Prof. KELLAND.)—ABC is a triangle inscribed in a circle; from any point O in the plane of the triangle perpendiculars are drawn to OA, OB, OC, meeting BC, CA, AB in D, E, F: prove (1) that the points D, E, F lie on a straight line, which (2), when O is on the circumference, passes through the centre of the circle.

Solution by J. J. WALKER, M.A.



1. Let OD meet CA in G, while OF, OB meet AD in H, K. From the equality of the angles AOD, AOG, BOE, COF, it is readily verified that the pencils O(FABD), O(AEGC) are equiangular; therefore, as regards AR, the pencil O(FABD) = D(AEGC), and therefore O(AKDH), or F(AKDH), or D(BAFO) = D(BAEO). Hence D, E, F are collinear.



2. Let K be the centre of the circle, which AK, CK meet in A′, C′, the former line meeting BC in P. Then

BD : DC = OB . BA′ : OC . CA′,

and

BP : PC = AB . BA′ : AC . CA′;

therefore

$$\frac{BD}{BP} : \frac{CD}{CP} = OB.AC : OC.AB.$$

Hence, by Ptolemy's theorem,

$$\frac{BD}{BC} : \frac{PD}{PC} = OB.CA : OA.BC.$$

Similarly, if BK meets CA in Q,

$$\frac{AE}{AC} : \frac{QE}{QC} = OA.BC : OB.CA$$
$$= \frac{PD}{PC} : \frac{BD}{BC}.$$

Hence, the points AKP, BKQ being collinear, D, K, E are also collinear. Similarly it may be shown that D, K, F are collinear.

3. By coordinates both parts of the theorem may be proved thus:—

Employing rectangular axes through O as origin, let $\alpha\beta$, $\alpha'\beta'$, $\alpha''\beta''$ be the coordinates of A, B, C; x_1y_1, x_2y_2, x_3y_3 those of D, E, F respectively; then we have

$$x_1 = \frac{(\alpha''\beta' - \alpha'\beta'')\,\beta}{\alpha(\alpha'-\alpha'') + \beta(\beta'-\beta'')}, \quad y_1 = \frac{(\alpha'\beta'' - \alpha''\beta')\,\alpha}{\alpha(\alpha'-\alpha'') + \beta(\beta'-\beta'')},$$

$$x_2 = \frac{(\alpha\beta'' - \alpha''\beta)\,\beta'}{\alpha'(\alpha''-\alpha) + \beta'(\beta''-\beta)}, \quad y_2 = \frac{(\alpha''\beta - \alpha\beta'')\,\alpha'}{\alpha'(\alpha''-\alpha) + \beta'(\beta''-\beta)},$$

$$x_3 = \frac{(\alpha'\beta - \alpha\beta')\,\beta''}{\alpha''(\alpha-\alpha') + \beta''(\beta-\beta')}, \quad y_3 = \frac{(\alpha\beta' - \alpha'\beta)\,\alpha}{\alpha''(\alpha-\alpha') + \beta''(\beta-\beta')}.$$

From these values, we obtain $y_1(x_2 - x_3) = \alpha(\alpha''\beta' - \alpha'\beta'')$

$$\times \frac{\left[\beta\beta'\beta''\left\{\alpha(\beta'-\beta'') + \alpha'(\beta''-\beta) + \alpha''(\beta-\beta')\right\} + \alpha'\alpha''\beta(\alpha'\beta'' - \alpha''\beta')\,n + \dots\right]}{\left\{\alpha(\alpha'-\alpha'') + \beta(\beta'-\beta'')\right\}\left\{\alpha'(\alpha''-\alpha) + \beta'(\beta''-\beta)\right\}\left\{\alpha''(\alpha-\alpha') + \beta''(\beta-\beta')\right\}}$$

Since the fraction-factors are symmetrical, it follows that $y_1(x_2-x_3)$, $y_2(x_3-x_1)$, and $y_3(x_1-x_2)$ are proportional to $\alpha(\alpha'\beta''-\alpha''\beta')$, $\alpha'(\alpha''\beta-\alpha\beta'')$, and $\alpha''(\alpha\beta'-\alpha'\beta)$, and that D, E, F are therefore collinear, whatever be the position of O in the plane of ABC.

If now the circle circumscribing ABC passes through O, the relation

$$(\alpha'\beta''-\alpha''\beta')(\alpha^2+\beta^2)+(\alpha''\beta-\alpha\beta'')(\alpha'^2+\beta'^2)+(\alpha\beta'-\alpha'\beta)(\alpha''^2+\beta''^2)=0$$

holds; and, if (ξ, η) be the centre of the circle, we have

$$\xi=\frac{\beta'(\alpha^2+\beta^2)-\beta(\alpha'^2+\beta'^2)}{2(\alpha\beta'-\alpha'\beta)},\quad \eta=\frac{\alpha(\alpha'^2+\beta'^2)-\alpha'(\alpha^2+\beta^2)}{2(\alpha\beta'-\alpha'\beta)}.$$

If these values for ξ, η, and those for x_1, y_1, x_2, y_2 given above, are substituted in $(x_1-x_2)\eta-(y_1-y_2)\xi+x_1y_2-x_2y_1$, the numerator of the result reduces identically to

$$\{(\alpha^2-\beta^2)\alpha'\beta'+(\alpha'^2-\beta'^2)\alpha\beta\}$$
$$\times\{(\alpha'\beta''-\alpha''\beta')(\alpha^2+\beta^2)+(\alpha''\beta-\alpha\beta'')(\alpha'^2+\beta'^2)+(\alpha\beta'-\alpha'\beta)(\alpha''^2+\beta''^2)\},$$

which proves that D, E are on a diameter of the circle.

5114. (By the EDITOR.)—Prove that if the equation $\sin^{-1}hx+\sin^{-1}ky=\alpha$ be referred to axes having any angle of ordination ω, it will represent an ellipse; and find the principal diameters of the curve.

Solution by Professor TOWNSEND, F.R.S.; Professor WOLSTENHOLME, M.A.; Professor EVANS, M.A.; *and others.*

Taking the cosines of both sides, we have

$$(1-h^2x^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{2}}-hk\,xy=\cos\alpha,$$

and this equation, when cleared of radicals, gives

$$h^2x^2+k^2y^2+2hk\cos\alpha\,xy=\sin^2\alpha,$$

the equation of an ellipse, the squared reciprocals of whose semiaxes a and b are given, by virtue of the two fundamental invariants of the coefficients, by the relations

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{h^2+k^2-2hk\cos\alpha\cos\omega}{\sin^2\alpha\sin^2\omega},\quad \frac{1}{a^2b^2}=\frac{h^2k^2}{\sin^2\alpha\sin^2\omega};$$

and their lines of direction by the equation

$$h(h\cos\omega-k\cos\alpha)x^2+(h^2-k^2)xy-k(k\cos\omega-h\cos\alpha)y^2=0.$$

5091. (By T. COTTERILL, M.A.)—If two points are such that the tangents to a parabola from one point are perpendicular to the tangents from the other; prove that (1) the points correspond homologically, the focus and directrix of the parabola being the centre and axis of homology; (2) besides

the focus there is another point which, taken as the focus of a conic, remains the focus of the transformed conic. [This is remarkable, being a limiting case of Walker's transformation in which his construction fails.]

Solution by the Rev. F. D. Thomson, M.A.

More generally, if the tangents to a conic from A meet in a fixed straight line in two points, and from these points the other tangents are drawn meeting in B, A and B are obviously corresponding points in a homographic system, since to each point there is one and one only corresponding point. Every point on the straight line is its own conjugate, as is also the pole of the straight line. The line joining two conjugates passes through the pole, and the extremities of any chord through the pole are conjugate points. In the question the focus is the pole and the straight line is the directrix of the parabola.

The particular case may be solved analytically as follows :—

The equations to any two tangents to the parabola $y^2 = 4ax$ are of the form $\lambda^2 x - \lambda y + a = 0, \quad \mu^2 x - \mu y + a = 0$;

and the perpendicular tangents are

$$x + \lambda y + a\lambda^2 = 0, \quad x + \mu y + a\mu^2 = 0.$$

Hence A and B are given by

$$x : y : a = 1 : \lambda + \mu : \lambda\mu, \qquad x : y : a = \lambda\mu : -(\lambda + \mu) : 1;$$

and if A lie on the curve $F(x, y, a) = 0$, B lies on $F(a, -y, x) = 0$.

Thus the parabola $\qquad (x+a)^2 + y^2 = (x+p)^2$

becomes $\qquad (a+x)^2 + y^2 = \left(a + \frac{px}{a}\right)^2,$

another parabola with the same focus.

5055. (By Prof. Lloyd Tanner, M.A.)—If A, B, a, b are points on a conic, and Aa is parallel to the tangent at B, and Bb to the tangent at A; prove that (1) ab is parallel to AB; and (2) if the conic be a parabola, the distance of the point of intersection of the tangents at A, B from ab, is five times its distance from AB.

I. *Solution by* Arthur Cohen, Q.C.

1. If the tangents at A and B be taken as axes of x and y, the equations of the conic and the line AB are

$$\left(\frac{x}{l} + \frac{y}{m} - 1\right)^2 = \frac{xy}{n^2}, \quad \frac{x}{l} + \frac{y}{m} - 1 = 0 \ldots\ldots\ldots\ldots (1).$$

Since Aa is parallel to the tangent at B, or to the axis of y, its equation is $x - l = 0$. The coordinates of the point a are therefore $x = l$, and for y the value, $y = lm^2n^{-2}$, that satisfies the equation to the conic when $x = l$.

Thus the coordinates of a, b are $x = l, \; y = \frac{m^2 l}{n^2}; \quad y = m, \; x = \frac{l^2 m}{n^2}$.

The equation to ab is therefore

$$Y - m = \frac{m - \dfrac{m^2 l}{n^2}}{\dfrac{l^2 m}{n^2} - l}\left(X - \frac{l^2 m}{n^2}\right) = -\frac{m}{l}\left(X - \frac{l^2 m}{n^2}\right);$$

therefore $$\frac{X}{l} + \frac{Y}{m} = 1 + \frac{lm}{n^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Therefore the line ab is parallel to the line AB.

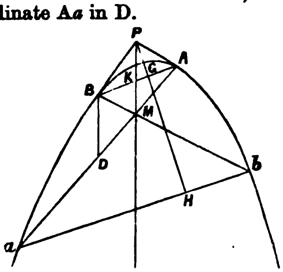
2. If the conic be a parabola, the terms of the second degree in its equation must constitute a perfect square, and therefore

$$\frac{4}{l^2 m^2} = \left(\frac{2}{lm} - \frac{1}{n^2}\right)^2; \text{ therefore } \frac{lm}{n^2} = 4.$$

Now the distances of the origin of coordinates from the lines (1) and (2) evidently bear to one another the ratio of 1 to $1 + \frac{lm}{n^2}$, or 1 to 5. In other words, the distance of the intersection of the tangents at A and B from ab is five times its distance from AB.

II. *Solution by* S. A. Renshaw.

1. Let Aa, Bb meet in M, and the tangents at A and B in P; and, in the case of the parabola, join PM, meeting AB and ab in K and N, and through B draw a diameter meeting its ordinate Aa in D.



Then in *any conic*, by Hamilton's Theorem, AM . Ma : BM . Mb = BP : AP^2; that is, since PAMB is a parallelogram, aM : bM = AM : BM; therefore the triangles ABM, abm are similar, and the angle Mba = MBA; hence AB and ab are parallel.

2. Again, in the parabola, since AD is an ordinate to the diameter BD, and BPMD, BPAM are parallelograms, Aa = 2AD = 4AM; therefore aM = 3AM; and, by similar triangles, NM = 3KM; consequently NP = 5PK; and, if PGH be at right angles to ab and AB, it follows that PH = 5PG.

III. *Solution by the* Proposer, R. Tucker, M.A., *and others.*

1. Refer the conic to tangents at A, B as axes; then its equation, and he equation of the chord of contact AB, are

$$\left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right)^2 - 2\lambda xy = 0, \quad \frac{x}{\alpha} + \frac{y}{\beta} - 1 = 0 \ldots\ldots\ldots (1, 2),$$

where λ is indeterminate.

The equation to Aa being $y = \beta$, we find, by substitution in (1), that the coordinates of a are $2\lambda\alpha^2\beta$, β. Similarly, the coordinates of b are α, $2\lambda\alpha\beta^2$. Hence the equation to ab is

$$\begin{vmatrix} x, & y, & 1 \\ \alpha, & 2\lambda\alpha\beta^2, & 1 \\ 2\lambda\alpha^2\beta, & \beta, & 1 \end{vmatrix} = 0, \text{ or } \frac{x}{\alpha} + \frac{y}{\beta} - 1 - 2\lambda\alpha\beta = 0,$$

so that ab is parallel to AB (2).

2. The condition that (1) should represent a parabola is

$$\left(\frac{1}{\alpha\beta}-\lambda\right)^2=\frac{1}{\alpha^2}\cdot\frac{1}{\beta^2},$$

whence, since λ is not zero, $2\lambda\alpha\beta = 4$.

Thus the equation to ab may be written

$$\frac{x}{5\alpha}+\frac{y}{5\beta}-1=0,$$

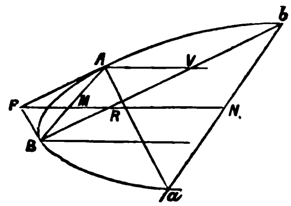
which proves the second part of the theorem.

IV. *Solution by* Rev. F. D. THOMSON M.A.; Prof. EVANS, M.A.; R. W. GENESE, M.A.; *and others.*

1. This is a case of Pascal's Theorem. The opposite sides AA, Bb; Aa, BB; ab, BA intersect on the line at infinity.

2. In the parabola draw the diameters through AB and point P, where the tangents meet as in the figure.

Then, since AV bisects Bb, and BR = PA = RV, therefore Bb = 4BR; therefore MN = 4MR = 4PM; therefore PN = 5PM.



5064. (By C. LEUDESDORF, M.A.)—Solve the simultaneous equations

$$x(y+z^{-1})=a,\quad y(z+x^{-1})=b,\quad z(x+y^{-1})=c;$$

considering in particular the case where the quantities a, b, c are connected by the equations $1+a+ac=0,\ 1+c+bc=0$.

Solution by OMAKYA CHAKRAVARTI; NARENDRA LAL DEY; ST. JOHN STEPHEN; Prof. NASH; *and many others.*

We have $\qquad xyz = az-x = bx-y = cy-z;$

whence $\qquad \dfrac{x}{1+a+ac}=\dfrac{y}{1+b+ab}=\dfrac{z}{1+c+bc}=\text{R say};$

therefore $\quad \text{R}^3(1+a+ac)(1+b+ab)(1+c+bc)$

$$=\text{R}\,[a(1+c+bc)-(1+a+ac)]=\text{R}\,(abc-1);$$

therefore $\quad x=\pm\left(\dfrac{(abc-1)(1+a+ac)}{(1+b+ba)(1+c+bc)}\right)^{\frac12},\quad y=\pm\ \&\text{c.},\quad z=\pm\ \&\text{c.}$

Now when $1+a+ac=0,\ 1+c+bc=0$ (and therefore also $1+b+ab=0$), we have $x=\dfrac{0}{0}$, and similarly for y and z. But since

$$1+a+ac=(1+a)(1+c)-c=ac\,.\,bc-c=c\,(abc-1),$$

$$1+b+ab=a\,(abc-1),\quad 1+c+bc=b\,(abc-1),$$

we find (although $abc-1$ also $=0$)

$$x = \pm\left(\frac{1\,.\,c}{a\,.\,b}\right)^{\frac{1}{2}} = \pm\left(\frac{c^2}{abc}\right)^{\frac{1}{2}} = \pm c, \quad y = \pm a, \quad z = \pm b.$$

But these values will not satisfy the original equations. Under the given conditions, in fact, the equations are not independent. For, since

$$a = -\frac{1}{1+c}, \text{ and } b = -\frac{1+c}{c},$$

the first and second equations become

$$xyz + cxyz + x + cx + z = 0, \quad cxyz + x + cx + cy = 0,$$

and from these, by subtraction, we at once obtain equation (3).

5115. (By C. W. MERRIFIELD, F.R.S.)—Of the three rectangles between opposite edges of a tetrahedron, prove that any two are together greater than the third.

I. *Solution by* Prof. TOWNSEND, Prof. WOLSTENHOLME, *and others.*

If A, B, C, D be the four vertices of the tetrahedron, and A′, B′, C′, D′ their four projections from any point O on the sphere passing through them upon the tangent plane at the opposite point; then, since

$$\text{BC}\,.\,\text{AD} : \text{B}'\text{C}'\,.\,\text{A}'\text{D}' = \text{CA}\,.\,\text{BD} : \text{C}'\text{A}'\,.\,\text{B}'\text{D}' = \text{AB}\,.\,\text{CD} : \text{A}'\text{B}'\,.\,\text{C}'\text{D}'$$
$$= \text{OA}^{\frac{1}{2}}\,.\,\text{OB}^{\frac{1}{2}}\,.\,\text{OC}^{\frac{1}{2}}\,.\,\text{OD}^{\frac{1}{2}} : \text{OA}'^{\frac{1}{2}}\,.\,\text{OB}'^{\frac{1}{2}}\,.\,\text{OC}'^{\frac{1}{2}}\,.\,\text{OD}'^{\frac{1}{2}},$$

and since the four points A′, B′, C′, D′ are not concyclic, as the four A, B, C, D are not complanar, therefore &c.

II. *Solution by the* PROPOSER.

Let V be the volume, and R the radius of the circumscribed sphere, it is easily shown (see CARNOT, *Mémoire sur la relation .... de cinq points*, &c.) that

$$9\text{R}^2\text{V}^2 = (ap+bq+cr)(-ap+bq+cr)(ap-bq+cr)(ap+bq-cr),$$

where a, b, c are the edges of the base, and p, q, r the other three conterminous edges. Neither of these factors can vanish without the tetrahedron vanishing too: hence they can none of them change sign as a, b, c, p, q, r vary. But in the case where

$$a = b = c = p = q = r,$$

they are all positive. Hence the theorem.

5088. (By R. W. GENESE, M.A.)—Find the locus of the poles of normals to a conic.

Solution by R. E. Riley, B.A.; H. T. Gerrans; *and others.*

The normal at (ϕ) is $ax\sec\phi - by\,\text{cosec}\,\phi = c^2$; then, if (ξ, η) is the pole of this, $\frac{x\xi}{a^2}+\frac{y\eta}{b^2}=1$ must be identical with it; hence, eliminating ϕ, the equation to the locus is $\qquad \frac{a^6}{\xi^2}+\frac{b^6}{\eta^2}=c^4.$

In the parabola the locus is $(x+2a)\,y^2+4a^3=0$.

5095. (By J. J. Walker, M.A.)—If (ξ, η) is the pole of the chord PQ, normal at P to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that the equation to the normal at Q is

$$\frac{a^2}{b^2-\eta^2}\,\frac{x}{\xi}+\frac{b^2}{a^2-\xi^2}\,\frac{y}{\eta}=\frac{c^4}{b^2\xi^2+a^2\eta^2}.$$

I. *Solution by* E. Rutter, R. E. Riley, B.A., H. T. Gerrans, *and others.*

If P is the point ϕ, then the following equations are identical,

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}=1, \qquad ax\sec\phi - by\,\text{cosec}\,\phi = a^2-b^2.$$

Thus
$$\xi = \frac{a^3\sec\phi}{a^2-b^2},\quad \eta = -\frac{b^3\,\text{cosec}\,\phi}{a^2-b^2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now Q is

$$\frac{\left\{a\,(a^2-b^2)^2-ab^4\,\text{cosec}^2\phi\right\}\sec\phi}{a^4\sec^2\phi+b^4\,\text{cosec}^2\phi},\quad \frac{\left\{b\,(a^2-b^2)^2-a^4b\sec^2\phi\right\}\text{cosec}\,\phi}{a^4\sec^2\phi+b^4\,\text{cosec}^2\phi},$$

or, by (1),
$$\frac{b^2-\eta^2}{b^2\xi^2+a^2y^2}\cdot\xi c^2,\quad -\frac{a^2-\xi^2}{b^2\xi^2+a^2y^2}\cdot\eta c^2.$$

Hence, the normal at (x', y') being $\frac{a^2x}{x'}-\frac{b^2y}{y'}=c^2$, the normal at Q is as stated in the question.

II. *Solution by* R. Tucker, M.A.; R. Battle; *and others.*

Let ϕ be the eccentric angle of the point Q, then writing

$$\lambda' \text{ for } ax,\qquad \mu' \text{ for } -by,\qquad \lambda \text{ for } \frac{\xi}{a},\qquad \mu \text{ for } \frac{\eta}{b},$$

we have, since Q is on the normal,

$$\lambda'\sin\phi+\mu'\cos\phi = c^2\sin\phi\cos\phi\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and since Q is also on the chord of contact PQ,

$$\lambda\cos\phi+\mu\sin\phi = 1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Multiplying (1) and (2), we have

$$\lambda\mu^2\cos^2\phi+\lambda'\mu\sin^2\phi+(\lambda\lambda'+\mu\mu'-c^2)\sin\phi\cos\phi = 0\ldots\ldots\ (3);$$

and squaring (2),

$$(\lambda^2-1)\cos^2\phi+(\mu^2-1)\sin^2\phi+2\lambda\mu\sin\phi\cos\phi = 0\ldots\ldots\ldots(4).$$

Hence, eliminating ϕ, we get

$$\frac{\lambda\mu'}{\lambda^2-1} = \frac{\lambda'\mu}{\mu^2-1} = \frac{\lambda\lambda'+\mu\mu'-c^2}{2\lambda\mu};$$

whence, substituting, we get

$$\frac{\xi y}{a^2-\xi^2} = -\frac{\eta x}{b^2-y^2} = \frac{x\xi-\eta y-c^2}{2\xi\eta},$$

or
$$\frac{\eta y}{\eta\dfrac{a^2-\xi^2}{\xi}} = -\frac{\xi x}{\xi\dfrac{b^2-y^2}{\eta}} = \frac{x\xi-\eta y-c^2}{2\xi\eta} = -\frac{c^2}{\dfrac{a^2y}{\xi}+\dfrac{b^2\xi}{\eta}};$$

therefore
$$\frac{a^2}{b^2-\eta^2}\cdot\frac{x}{\xi}+\frac{b^2}{a^2-\xi^2}\cdot\frac{y}{\eta} = \frac{c^4}{a^2y^2+b^2\xi^2}.$$

III. *Solution by* J. L. McKenzie, B.A.

Let ϕ, ϕ' be the eccentric angles of P and Q; then, as in the first Solution, we have $\cos\phi = \dfrac{a^3}{c^2\xi}$, $\sin\phi = -\dfrac{b^3}{c^2\eta}$ ....(1).

But when (ξ, η) is the pole of the chord through ϕ, ϕ', it is easily proved that $\sin\phi\sin\phi' = \dfrac{b^2(a^2-\xi^2)}{b^2\xi^2+a^2\eta^2}$, $\cos\phi\cos\phi' = \dfrac{a^2(b^2-\eta^2)}{b^2\xi^2+a^2\eta^2}$ ....(2).

The normal at Q is $\dfrac{ax}{\cos\phi'} - \dfrac{by}{\sin\phi'} = c^2$; and substituting the values of $\sin\phi'$ and $\cos\phi'$ from (2) and (1), we get

$$\frac{a^2x}{(b^2-\eta^2)\xi}+\frac{b^2y}{(a^2-\xi^2)\eta} = \frac{c^4}{b^2\xi^2+a^2\eta^2}.$$

[The relations (2) may be readily obtained thus:—

Since the line $\dfrac{\xi x}{a^2}+\dfrac{\eta y}{b^2} = 1$ passes through ϕ, we have

$$\frac{\xi\cos\phi}{a}+\frac{\eta\sin\phi}{b} = 1;$$

which may be put into either of the forms

$$\sin^2\phi\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}\right)-\frac{2\eta}{b}\sin\phi+1-\frac{\xi^2}{a^2} = 0,$$

$$\cos^2\phi\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}\right)-\frac{2\xi}{a}\cos\phi+1-\frac{\eta^2}{b^2} = 0.$$

The product of the roots in the first form is $\sin\phi\sin\phi'$, in the second $\cos\phi\cos\phi'$.]

5107. (By R. Tucker, M.A.)—A normal chord PQ to an ellipse is drawn, find the locus of the intersection of the tangent at P with the normal at Q.

Solution by the PROPOSER.

With the notation employed in the solution of Question 5095 (p. 99 of this Vol.), the equation to the tangent, which is in this case the perpendicular to PQ through (ξ, η) is $a^2\dfrac{x}{\xi} - b^2\dfrac{y}{\eta} = c^2$ ....(1).

From the condition that (1) is a tangent (or otherwise) we readily get

$$\frac{a^6}{\xi^2} + \frac{b^6}{\eta^2} = c^4 \quad \text{..............................(2).}$$

[This result at once gives the locus sought in Question 5088].

The required locus is found by eliminating ξ, η between (1), (2), and the equation of Question 5095,

$$\frac{a^2}{b^2-\eta^2}\cdot\frac{x}{\xi} + \frac{b^2}{a^2-\xi^2}\cdot\frac{y}{\eta} = \frac{c^4}{a^2y^2+b^2\xi^2} \quad \text{..................(3).}$$

From (1) and (2), ξ and η can be found; putting these values in (3), we should get the locus required.

[Mr. TUCKER states, however, that after many attempts he has not been able to obtain a clear result.]

5116. (By T. COTTERILL, M.A.)—The straight lines connecting n points in a plane intersect again in N points, N being three times the number of the combinations of n things taken four together. If the n points are on a conic, show that the N points are made up of self-conjugate triads to the conic, so that the number of these points outside the conic is twice the number inside.

I. *Solution by* E. B. ELLIOTT, M.A.

Take any four of the n points as the vertices of an inscribed quadrangle. The six sides of this quadrangle intersect by pairs in three of the N points, the vertices of the harmonic triangle of the quadrangle, and therefore a self-conjugate triad. Of these three, (1) at least one must be within the conic—for, if two be without, the third, which is the pole of their connector, must be within; and (2) at least two must be without it—for, if one be within, the other two lie on its polar, which is entirely without. Thus, of the three, one must be within and two without the conic.

In this way the whole of the N points are obtained (each occurring once only) in self-conjugate sets of three, two of each set without and one within the conic.

II. *Solution by* Professor WOLSTENHOLME, M.A.

Taking any four of the n points, the lines joining them, two and two, intersect again in three points, which are the corners of a triangle, self-conjugate to the conic, and of which therefore one is within and two without the conic. [For, if A, B, C be the three points, if A be within, every point on its polar,—and therefore B and C,—must be without; and

they cannot all three be without, since the pole of any line altogether without, the conic must be within.]

Since to every combination of four we get three new points of intersection, the whole number will be three times the number of combinations of n things taken four together, or $\frac{1}{8}n(n-1)(n-2)(n-3)$.

[This result was set in the Senate House some years ago, and is one of the examples in TODHUNTER'S *Algebra;* but this question suggests this much simpler way of getting at the result. The direct way would be thus:— The number of joining lines is $\frac{1}{2}n(n-1)$ $(=m)$, and the whole number of intersections would be $\frac{1}{2}m(m-1)$; but of these $\frac{1}{2}(n-1)(n-2)$ are lost at each of the n points, so that the number of new intersections is

$$\tfrac{1}{2}m(m-1)-\tfrac{1}{2}n(n-1)(n-2),$$

or $$\tfrac{1}{2}n(n-1)\left\{\tfrac{1}{4}n(n-1)-\tfrac{1}{2}-(n-2)\right\} \equiv \tfrac{1}{8}n(n-1)(n-2)(n-3)].$$

3999 (By Dr. HART.)—Find three numbers whose sum is a cube, the sum of their squares a cube, and the sum of their cubes a square.

Solution by the PROPOSER.

Let ax^3, bx^3, cx^3 be the numbers; then we must have

$$(a+b+c)\,x^3 = \text{cube}, \quad \text{or} \quad a+b+c = \text{cube} \;\ldots\ldots\ldots\ldots(1),$$

$$(a^2+b^2+c^2)\,x^6 = \text{cube}, \quad \text{or} \quad a^2+b^2+c^2 = \text{cube} \;\ldots\ldots\ldots\ldots(2),$$

and $$(a^3+b^3+c^3)\,x^9 = \square = x^{10}, \quad \text{whence} \quad x = a^3+b^3+c^3.$$

Since (1) and (2) are to be cubes, their product $(a+b+c)(a^2+b^2+c^2)$ $=$ cube. Let this $=(a+b-c)^3$. Performing the operations indicated, we have $$(b-2c)\,a^2+(b^2-3bc+c^2)\,a = 2b^2c-bc^2+c^3,$$

and dividing by $b-2c$, $$a^2+\frac{b^2-3bc+c^2}{b-2c}\,a = \frac{2b^2c-bc^2+c^3}{b-2c};$$

therefore $$a = -\frac{b^2-3bc+c^2}{2(b-2c)} \pm \frac{(b^4+2b^3c-9b^2c^2+6bc^3-7c^4)^{\frac{1}{2}}}{2(b-2c)};$$

therefore we must make $b^4+2b^3c-9b^2c^2+6bc^3-7c^4 = \square$ ……………(3), which is so if $b=2c$. Let $b=2c+d$; then, by substitution in (3), we have $$c^4+26c^3d+27c^2d^2+10cd^3+d^4 = \square;$$

let this $=(c^2+13cd-d^2)^2$. Reducing, we find $d=\frac{1}{9}(35c)$. Take $c=9$, then $d=35$, $b=53$; therefore $a=-63$ or 63; or, if we put the above $=(c^2+13cd-71d^2)^2$, we shall have $d=\frac{116}{315}c$. Take $c=315$, then $d=116$, $b=746$; therefore $a=1855$. If these values be multiplied by 2, we have $a=3710$, $b=1492$, $c=630$. Using the first set, $x \equiv a^3+b^3+c^3 = 399653$; therefore

$$ax^3 = 63\,(399653)^3, \quad bx^3 = 53\,(399653)^3, \quad \text{and} \quad cx^3 = 9\,(399653)^3.$$

The sum of the numbers in the second set is $2916 = 9\,(18)^2$; and, multiplying each number by 2, the sum is $(18)^3 =$ cube. In this case, although the product of the two factors is a cube, each factor is not so, but can be made a cube by multiplication as above; and we may proceed thus in similar cases.

4164. (By H. MURPHY.)—If the polars of two fixed points A, B pass through two given points C, D, prove that the centre of the circle regarding which they are the polars ranges on the radical axis of two fixed circles.

I. *Solution by* S. FORDE, M.A.; C. LEUDESDORF, M.A.; *and others.*

Taking AB as axis of x, CD as axis of y, and

$$(x-\alpha)^2+(y-\beta)^2+2(x-\alpha)(y-\beta)\cos\omega = r^2$$

for the circle; if the polar of $A(h_1, 0)$ pass through $C(0, k_1)$,

$$\alpha(h_1-\alpha-\beta\cos\omega)+(\beta-k_1)\left[(h_1-\alpha)\cos\omega-\beta\right]+r^2=0 \text{ ...... (1)};$$

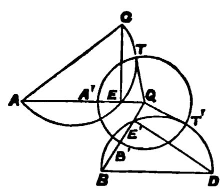
and if the polar of $B(h_2, 0)$ pass through $D(0, k_2)$, we have

$$\alpha(h_2-\alpha-\beta\cos\omega)+(\beta-k_2)\left[(h_2-\alpha)\cos\omega-\beta\right]+r^2=0 \text{ ...... (2)};$$

therefore $\alpha(h_1-h_2)+\beta\cos\omega(h_1-h_2)+\alpha\cos\omega(k_1-k_2)=0.$

And this is the radical axis of the two circles whose equations are found by writing x for α and y for β in (1) and (2).

II. *Solution by* Prof. EVANS, M.A., *the* PROPOSER, *and others.*



Let A be any position of the centre of the variable circle; draw AO, BO cutting this circle in A′, B′; and draw CE and DE′ cutting AO and BO perpendicularly in E and E′. Let OT, OT′ be tangents from A to the circles described on AC and BD as diameters.

Since CE is the polar of A, we have

$OA'^2 = OE \,.\, OA$; but $OE \,.\, OA = OT^2$;

therefore $OA' = OT'$.

Similarly, DE′ being the polar of B, $OB'^2 = OE' \,.\, OB$; and $OE' \,.\, OB = OT'^2$; therefore $OB' = OT$. Now $OA' = OB'$, and therefore $OT = OT'$; or, O is on the radical axis of the two fixed circles described on AC and BD as diameters.

5001. (By T. COTTERILL, M.A.)—To a system of five points on a conic, show that there is a point upon the conic such that the line through it parallel to the chord joining two of the points passes through the remaining intersection of the conic and circle through the remaining three points.

Solution by Prof. NASH, M.A.; C. LEUDESDORF, M.A.; *and others.*

To prove the property for an ellipse, let $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ be the eccentric angles of the points A, B, C, D, E.

The circle round CDE cuts the ellipse again in G, whose coordinates are

$$a\cos(\theta_3+\theta_4+\theta_5) - b\sin(\theta_3+\theta_4+\theta_5).$$

The equation of line through G, parallel to AB, is

$$y + b\sin(\theta_3+\theta_4+\theta_5) = -\cos\tfrac{1}{2}(\theta_1+\theta_2)\left\{x - a\cos(\theta_3+\theta_4+\theta_5)\right\};$$

and, if ϕ be the eccentric angle of the point P where it cuts the curve again, putting $x = a\cos\phi$, $y = b\sin\phi$, we obtain

$$\cos\left\{\phi - \tfrac{1}{2}(\theta_1+\theta_2)\right\} = \cos\left\{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5 - \tfrac{1}{2}(\theta_1+\theta_2)\right\};$$

therefore $\qquad \phi = 2n\pi + \theta_1+\theta_2+\theta_3+\theta_4+\theta_5;$

therefore P is a fixed point in the ten possible arrangements of the points A, B, C, D, E.

[A Solution by Dr. Hirst is given on p. 17 of this Volume of the *Reprint*.]

4892. (By W. S. B. Woolhouse, F.R.A.S.)—Three functional equations, $\phi_1(x, y, z) = 0$, $\phi_2(x, y, z) = 0$, $\phi_3(x, y, z) = 0$, are mutually dependent when

$$\begin{vmatrix} \dfrac{d\phi_1}{dx}, & \dfrac{d\phi_1}{dy}, & \dfrac{d\phi_1}{dz} \\ \dfrac{d\phi_2}{dx}, & \dfrac{d\phi_2}{dy}, & \dfrac{d\phi_2}{dz} \\ \dfrac{d\phi_3}{dx}, & \dfrac{d\phi_3}{dy}, & \dfrac{d\phi_3}{dz} \end{vmatrix} = 0.$$

Solution by Professor Wolstenholme, M.A.

This, in the general case of n functions, is a well-known proposition, and was set by me in the Senate House Examination of 1863.

Let there be n functions $V_1, V_2, \dots V_n$ of the variables $x_1, x_2, \dots x_m$; where m is not less than n: these functions will be connected by an equation, provided that

$$\left\|\begin{matrix} \dfrac{dV_1}{dx_1}, & \dfrac{dV_2}{dx_1}, & \dfrac{dV_3}{dx_1}, & \dots & \dfrac{dV_n}{dx_1} \\ \dfrac{dV_1}{dx_2}, & \dfrac{dV_2}{dx_2}, & \dots & \dots & \dfrac{dV_n}{dx_2} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{dV_1}{dx_m}, & \dfrac{dV_2}{dx_m}, & \dots & \dots & \dfrac{dV_n}{dx_m} \end{matrix}\right\| = 0, \quad \text{or} \quad \left\|\begin{matrix} \dfrac{dV_1}{dx_1}, & \dfrac{dV_1}{dx_2}, & \dots & \dfrac{dV_1}{dx_m} \\ \dfrac{dV_2}{dx_1}, & \dfrac{dV_2}{dx_2}, & \dots & \dfrac{dV_2}{dx_m} \\ \dots & \dots & \dots & \dots \\ \dfrac{dV_n}{dx_1}, & \dfrac{dV_n}{dx_2}, & \dots & \dfrac{dV_n}{dx_m} \end{matrix}\right\| = 0.$$

For suppose an equation $\phi(V_1, V_2, \dots V_n) = 0$ to exist, then we shall have the system of equations

$$\frac{d\phi}{dV_1}\frac{dV_1}{dx_1} + \frac{d\phi}{dV_2}\frac{dV_2}{dx_1} + \dots = 0,$$

$$\frac{d\phi}{dV_1}\frac{dV_1}{dx_2} + \frac{d\phi}{dV_2}\frac{dV_2}{dx_2} + \dots = 0,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\frac{d\phi}{dV_1}\frac{dV_1}{dx_m} + \frac{d\phi}{dV_2}\frac{dV_2}{dx_m} + \dots = 0,$$

leading at once to the determinant, or determinants, stated.

The proof might also be given as follows:—Suppose such increments $dx_1, dx_2, \ldots dx_m$ be given to $x_1, x_2, \ldots x_m$, that $V_1, V_2, \ldots V_{n-1}$ may be unaltered; then, if a relation exist between $V_1, V_2, \ldots V_n$, V_n must also be unaltered, and the increment must satisfy the equations

$$\frac{dV_1}{dx_1}dx_1+\frac{dV_1}{dx_2}dx_2+\frac{dV_1}{dx_3}dx_3+\ldots\ldots+\frac{dV_1}{dx_m}dx_m = 0,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

leading at once to the determinant, or determinants, in the second form.

Example: $$(a^2+b^2+c^2)(x^2+y^2+z^2)$$
$$=(ax+by+cz)^2+(bz-cy)^2+(cx-az)^2+(ay-bx)^2.$$

Hence
$$\begin{vmatrix} a, & 0, & x, & 0, & -z, & y \\ b, & 0, & y, & z, & 0, & -x \\ c, & 0, & z, & -y, & x, & 0 \\ 0, & x, & a, & 0, & c, & -b \\ 0, & y, & b, & -c, & 0, & a \\ 0, & z, & c, & b, & -a, & 0 \end{vmatrix} = 0,$$

the example given in the Senate House, 1863 (with a slight misprint).

4429. (By W. Siverly.)—Three circles, of radii R_1, R_2, R_3, touch each other externally; a circle (of radius r_1) is drawn touching these three circles externally; then another (of radius r_2) is drawn to touch R_1, R_2, and r_1; then another touching R_1, R_2, and r_2; and so on. Find (1) the value of r_n in terms of R_1, R_2, R_3; and (2) the sum of the areas of the circles r_1, r_2, r_3, &c. to infinity.

Solution by Professor Evans, M.A.

1. Let ρ_1, ρ_2, ρ_3 be the radii of any three circles tangent to one another externally, and ρ the radius of a fourth circle tangent externally to each of them. Let A, B, C, O be the centres of these four circles; then the triangles ABC, AOB, AOC give

$$\cos^2\tfrac{1}{2}BAC = \frac{\rho_1(\rho_1+\rho_2+\rho_3)}{(\rho_1+\rho_2)(\rho_1+\rho_3)},\quad \cos^2\tfrac{1}{2}BAO = \frac{\rho_1(\rho+\rho_1+\rho_2)}{(\rho+\rho_1)(\rho_2+\rho_3)},$$

$$\cos^2\tfrac{1}{2}CAO = \frac{\rho_1(\rho+\rho_1+\rho_3)}{(\rho+\rho_1)(\rho_1+\rho_3)},\quad \sin^2\tfrac{1}{2}BAO = \frac{\rho\rho_2}{(\rho+\rho_1)(\rho_1+\rho_2)},$$

and $$\sin^2\tfrac{1}{2}CAO = \frac{\rho\rho_3}{(\rho+\rho_1)(\rho_1+\rho_3)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since $\angle BAC = \angle BAO + \angle CAO$, we have

$$\cos\tfrac{1}{2}BAC = \cos\tfrac{1}{2}BAO\cos\tfrac{1}{2}CAO - \sin\tfrac{1}{2}BAO\sin\tfrac{1}{2}CAO \quad\ldots\ldots(2).$$

Substituting in (2) the values given by (1), we have,

$$(\rho+\rho_1)[\rho_1(\rho_1+\rho_2+\rho_3)]^{\frac{1}{2}} = \rho_1[(\rho+\rho_1+\rho_2)(\rho+\rho_1+\rho_3)]^{\frac{1}{2}} - \rho(\rho_2\rho_3)^{\frac{1}{2}};$$

whence $$\frac{1}{\rho} = \frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+2\left\{\frac{1}{\rho_1\rho_2}+\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\frac{1}{\rho_3}\right\}^{\frac{1}{2}} \quad\ldots\ldots\ldots\ldots(3).$$

Putting $a = \frac{1}{R_1} + \frac{1}{R_2}$, $b = \frac{1}{R_1 R_2}$, and applying equation (3) to the circles whose radii are r_{x+2} and r_{x+1}, we find

$$\frac{1}{r_{x+2}} = a + \frac{1}{r_{x+1}} + 2\left(b + \frac{a}{r_{x+1}}\right)^{\frac{1}{2}}, \text{ and } \frac{1}{r_{x+1}} = a + \frac{1}{r_x} + 2\left(b + \frac{a}{r_x}\right)^{\frac{1}{2}},$$

which, on being freed from radicals, become

$$\left(\frac{1}{r_{x+2}} - \frac{1}{r_{x+1}}\right)^2 - 2a\left(\frac{1}{r_{x+2}} + \frac{1}{r_{x+1}}\right) = 4b - a^2 \text{ .... (4).}$$

and

$$\left(\frac{1}{r_{x+1}} - \frac{1}{r_x}\right)^2 - 2a\left(\frac{1}{r_{x+1}} + \frac{1}{r_x}\right) = 4b - a^2 \text{ .... (5).}$$

Subtracting (5) from (4), and dividing by $\left(\frac{1}{r_{x+2}} - \frac{1}{r_{x+1}}\right)$, we get

$$\frac{1}{r_{x+2}} - \frac{2}{r_{x+1}} + \frac{1}{r_x} = 2a, \text{ or } \Delta^2 \frac{1}{r_x} = 2a.$$

Assuming $\frac{1}{r_x} = Ax^2 + Bx + C$, we obtain

$$\Delta^2 \frac{1}{r_x} = 2A = 2a, \text{ and } \frac{1}{r_0} = \frac{1}{R_3} = C.$$

Putting $x = 0$ in (5), we find

$$\frac{1}{r_1} = a + \frac{1}{R_3} + 2\left(b + \frac{a}{R_3}\right)^{\frac{1}{2}} = A + B + C;$$

therefore $$A = a, \quad B = 2\left(b + \frac{a}{R_3}\right)^{\frac{1}{2}}, \quad C = \frac{1}{R_3};$$

and hence $$\frac{1}{r_x} = ax^2 + 2x\left(b + \frac{a}{R_3}\right)^{\frac{1}{2}} + \frac{1}{R_3};$$

therefore $$\frac{1}{r_n} = an^2 + 2n\left(b + \frac{a}{R_3}\right)^{\frac{1}{2}} + \frac{1}{R_3}.$$

2. Restoring the values of a and b, we have

$$r_n = \frac{R_1 R_2 R_3}{(R_1 R_3 + R_2 R_3)\,n^2 + 2\left\{R_1 R_2 R_3 (R_1 + R_2 + R_3)\right\}^{\frac{1}{2}} n + R_1 R_2}.$$

Secondly, put $$p = \frac{2\left\{R_1 R_2 R_3 (R_1 + R_2 + R_3)\right\}}{R_1 R_3 + R_2 R_3},$$

$$q = \frac{R_1 R_2}{R_1 R_3 + R_2 R_3}, \text{ and } r = \frac{R_1 R_2 R_3}{R_1 R_3 + R_2 R_3}; \text{ then } r_n = \frac{r}{n^2 + pn + q},$$

and $$\pi\left(r_1^2 + r_2^2 + \dots + r_n^2\right)$$

$$= \pi r^2 \left\{\left(\frac{1}{1 + p + q}\right)^2 + \left(\frac{1}{4 + 2p + q}\right)^2 + \dots + \left(\frac{1}{n^2 + np + q}\right)^2\right\}.$$

4917. (By Professor Evans, M.A.)—Three circles touch one another externally, and within the space enclosed by this group three other circles

are inscribed touching one another, and each touching two circles of the first group; the xth group is similarly situated with reference to the group immediately preceding. If a, b, c, and a_x, b_x, c_x be the radii of the circles in the first and $(x+1)$th groups, prove that

$$\frac{1}{a_x}\pm\frac{1}{a}=\frac{1}{b_x}\pm\frac{1}{b}=\frac{1}{c_x}\pm\frac{1}{c}=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)p_x+\left(\frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc}\right)^{\frac{1}{2}}q_x,$$

where $(5+2\sqrt{x})^x\pm 1 = 3p_x+q_x\sqrt{6}$, p_x and q_x being both integers.

Solution by the PROPOSER.

Let $\dfrac{1}{a_x}+\dfrac{1}{b_x}+\dfrac{1}{c_x}=u_x$, and $\left(\dfrac{2}{a_xb_x}+\dfrac{2}{a_xc_x}+\dfrac{2}{b_xc_x}\right)^{\frac{1}{2}}=v_x$ ....(1);

then [see *Reprint*, Vol. XXIV., p. 109] we have

$$\frac{1}{a_x}=-\frac{1}{a_{x-1}}+2u_{x-1}+2v_{x-1},\qquad \frac{1}{b_x}=-\frac{1}{b_{x-1}}+2u_{x-1}+2v_{x-1},$$

and $$\frac{1}{c_x}=-\frac{1}{c_{x-1}}+2u_{x-1}+2v_{x-1}\ \text{........................(2)}.$$

Since $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=u,\quad \left(\frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc}\right)^{\frac{1}{2}}=v\ \text{..................(3)},$$

and $\dfrac{1}{a_1}=-\dfrac{1}{a}+2u+2v$, $\dfrac{1}{b_1}=-\dfrac{1}{b}+2u+2v$, $\dfrac{1}{c_1}=-\dfrac{1}{c}+2u+2v$ ...(4),

we have, by substitution,

$$\frac{1}{a_1}+\frac{1}{b_1}+\frac{1}{c_1}=u_1=5u+6v,\quad \left(\frac{2}{a_1b_1}+\frac{2}{a_1c_1}+\frac{2}{b_1c_1}\right)^{\frac{1}{2}}=v_1=4u+5v\ \text{...(5, 6)}.$$

Similarly, from (2), $u_x=5u_{x-1}+6v_{x-1}$, and $v_x=4u_{x-1}+5v_{x-1}$ ...(7).

By giving x the integral values 1, 2, 3, &c., we may obtain from (2) a series of equations that will enable us to find readily

$$\frac{1}{a_x}\pm\frac{1}{a}=\frac{1}{b_x}\pm\frac{1}{b}=\frac{1}{c_x}\pm\frac{1}{c}$$

$$=2\left[(u_{x-1}+v_{x-1})-(u_{x-2}+v_{x-2})+(u_{x-3}+v_{x-3})-\ \dots\ \pm(u+v)\right]\ \text{... (8)},$$

where the double sign $\pm$ is to be taken plus when x is odd and minus when x is even.

From the first of equations (7) $v_{x-1}=\frac{1}{6}(u_x-5u_{x-1})$, and therefore $v_x=\frac{1}{6}(u_{x+1}-5u_x)$. These values of v_{x-1} and v_x, substituted in equations (7), give $u_{x+1}-10u_x+u_{x-1}=0$; therefore

$$u_x-10u_{x-1}+u_{x-2}=0\ \text{..............................(9)}.$$

Equation (9) is an equation in finite differences whose solution is

$$u_x=C_1r_1^x+C_2r_2^x\ \text{..............................(10)},$$

where $r_1=5+2\sqrt{6}$ and $r_2=5-2\sqrt{6}$ are the roots of $x^2-10x+1=0$, and C_1 and C_2 are constants of integration. To determine these constants we observe that when $x=0$ and $x=1$, equation (10) gives

$$u=C_1+C_2,\quad\text{and}\quad u_1=5u+6v=C_1r_1+C_2r_2;$$

therefore $$C_1=\tfrac{1}{2}u+\tfrac{1}{4}v\sqrt{6},\quad\text{and}\quad C_2=\tfrac{1}{2}u-\tfrac{1}{4}v\sqrt{6}\ \text{...............(11)}.$$

Again, since $v_x=\frac{1}{6}(u_{x+1}-5u_x)$, we have

$$u_x+v_x=\tfrac{1}{6}(u_{x-1}+u_x)=\tfrac{1}{6}C_1(r_1^{x+1}+r_1^x+\tfrac{1}{6}C_2(r_2^{x+1}+r_2^x)\ \text{......(12)}.$$

Observing that $r_1+1 = 6+2\sqrt{6}$, and $r_2+1 = 6-2\sqrt{6}$, we may write (12) $u_x+v_x = \frac{1}{3}(6+2\sqrt{6})\,C_1 r_1^x + \frac{1}{3}(6-2\sqrt{6})\,C_2 r_2^x$ .... (13).

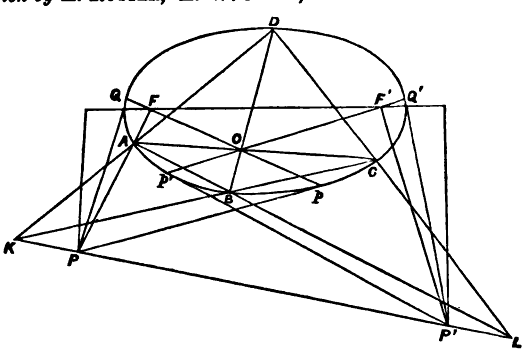
From (8) and (13) we have $\frac{1}{a_x} \pm \frac{1}{a} = \frac{1}{b_x} \pm \frac{1}{b} = \frac{1}{c_x} \pm \frac{1}{c}$

$$= \tfrac{1}{3}(6+2\sqrt{6})\,C_1\,[r_1^{x-1}-r_1^{x-2}+r_1^{x-3}- \dots \pm 1]$$
$$+ \tfrac{1}{3}(6-2\sqrt{6})\,C_2\,[r_2^{x-1}-r_2^{x-2}+r_2^{x-3}- \dots \pm 1]$$
$$= \frac{6+2\sqrt{6}}{3(r_1+1)}\,C_1(r_1^x \pm 1) + \frac{6-2\sqrt{6}}{3(r_2+1)}\,C_2(r_2^x \pm 1) = \tfrac{1}{3}C_1(r_1^x \pm 1) + \tfrac{1}{3}C_2(r_2^x \pm 1)$$
$$= \tfrac{1}{3}u\,[(5+2\sqrt{6})^x + (5-2\sqrt{6})^x \pm 2] + \frac{v}{2\sqrt{6}}\,[(5+2\sqrt{6})^x - (5-2\sqrt{6})^x]$$
$$= \tfrac{1}{3}\left\{(5+2\sqrt{6})^x + (5-2\sqrt{6})^x \pm 2\right\}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$
$$+ \frac{1}{2\sqrt{6}}\left\{(5+2\sqrt{6})^x - (5-2\sqrt{6})^x\right\}\left(\frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc}\right)^{\frac{1}{2}}$$
$$= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)p_x + \left(\frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc}\right)^{\frac{1}{2}} q_x,$$

where $(5+2\sqrt{6})^x \pm 1 = 3p_x + q_x\sqrt{6}$, p_x and q_x being both integers.

5068. (By S. A. Renshaw.)—If ABCD be a quadrilateral inscribed in a conic, and the opposite sides meet in K and L, and its diagonals in O; then, if KL be joined, cutting the directrix in P, prove that FO will be at right angles to FP.

Solution by E. Rutter, L. W. Jones, *and others.*



Let KL cut the directrices at P and P′. Now as O is the pole of KL, K the pole of LO, and L the pole of KO, therefore KL is the locus of the intersections of all tangents whose chords of contact pass through the pole O. Produce FO to meet the conic at Q, p; and F′O to meet it at Q′, p'. Because Qp, Q′p' are focal chords, P is the pole of Qp, and P′ of Q′p'. Therefore PFO, P′F′O are right angles. A

similar proof holds for the parabola and hyperbola. (See Davies's *Hutton*, Vol. II., pp. 133, 118, Cor. 2, &c.)

[This is a particular case of the theorem that if a straight line KL meet the directrix of a conic in P, and O be the pole of KL, then OP subtends a right angle at the corresponding focus. In the quadrilateral, O is the pole of KL with respect either to a circumscribed or an inscribed conic.]

4861. (By R. Tucker, M.A.)—Prove that the volume under the chordal segments of the altitudes of a triangle made by the inscribed circle is to the volume under the sides of the Pedal triangle as $16r$ is to the perimeter of the given triangle.

Solution by the Proposer.

Let δ be the chordal intercept on the altitude AD, then since (if O is the centre of the inscribed circle) the angle OAD $= \frac{1}{2}(C-B)$,

$$\delta^2 = 4\left\{r^2 - r^2 \operatorname{cosec}^2 \tfrac{1}{2}A \sin^2 \tfrac{1}{2}(C-B)\right\}$$

$$= 4r^2\left\{1 - \frac{\sin^2 \frac{1}{2}(C-B)}{\sin^2 \frac{1}{2}A}\right\} = 4r^2 \frac{\cos B \cdot \cos C}{\sin^2 \frac{1}{2}A};$$

therefore volume under chordal segments $(V) = \dfrac{8r^3 \cos A \cos B \cos C}{\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}$.

Volume under sides of pedal triangle $(V') = R^3 \sin 2A \sin 2B \sin 2C$;

therefore $$\frac{V}{V'} = \frac{r^3}{R^3 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C \sin A \sin B \sin C}$$

$$= \frac{64\Delta^4}{a^2b^2c^2 s^2 \sin A \sin B \sin C} = \frac{8\Delta}{s^2} = \frac{8r}{s} = \frac{16r}{a+b+c}.$$

4835. (By Artemas Martin.)—A speaks the truth a times out of b; B, c times out of d; and C, e times out of f. A relates a story; B tells it to C; C tells it to A; and A, forgetting that he first told the story himself, tells it as coming from C. What is the probability of the truth of the story?

Solution by the Proposer.

As A has forgotten that he first told the story himself he may be regarded as a new witness when he tells it the second time; we will, therefore, for convenience, call him D when he tells the story as coming from C.

According to the theory of *traditional* testimony laid down in Tod-

hunter's *Algebra*, 7th edition, page 472, art. 756, the story is true if all tell the truth, or if all lie, or if any two lie and the other two speak truth; and it is not true if any one lies and the other three speak truth, or if any three lie and the other one speaks truth.

Hence the story is true if all speak the truth, the chance of which is

$$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} \times \frac{a}{b} = \frac{a^2ce}{b^2df} = p_1;$$

or if all lie, the chance of which is

$$\left(1-\frac{a}{b}\right)^2 \left(1-\frac{c}{d}\right)\left(1-\frac{e}{f}\right) = p_2;$$

or if A and B lie and the others speak truth, the chance of which is

$$\frac{ae}{bf}\left(1-\frac{a}{b}\right)\left(1-\frac{c}{d}\right) = p_3;$$

or if A and C lie and the others speak truth, the chance of which is

$$\frac{ac}{bd}\left(1-\frac{a}{b}\right)\left(1-\frac{e}{f}\right) = p_4,$$

or if A and D lie and the others speak truth, the chance of which is

$$\frac{ce}{df}\left(1-\frac{a}{b}\right)^2 = p_5;$$

or if B and C lie and the others tell the truth, the chance of which is

$$\frac{a^2}{b^2}\left(1-\frac{c}{d}\right)\left(1-\frac{e}{f}\right) = p_6;$$

or if B and D lie and the others speak truth, the chance of which is

$$\frac{ae}{bf}\left(1-\frac{c}{d}\right)\left(1-\frac{a}{b}\right) = p_7;$$

or if C and D lie and the others tell the truth, the chance of which is

$$\frac{ac}{bd}\left(1-\frac{e}{f}\right)\left(1-\frac{a}{b}\right) = p_8.$$

The story is not true—

if A lies and the rest tell the truth, the chance of which put $= q_1$;
or if B lies and the rest speak truth, the chance of which put $= q_2$;
or if C lies and the rest tell the truth, the chance of which put $= q_3$;
or if D lies and the rest tell the truth, the chance of which put $= q_4$;
or if A, B, and C lie and D tells the truth, the chance of which put $= q_5$;
or if A, B, and D lie and C tells the truth, the chance of which put $= q_6$;
or if A, C, and D lie and B tells the truth, the chance of which put $= q_7$;
or if B, C, and D lie and A tells the truth, the chance of which put $= q_8$.

These sixteen chances include all possible cases; therefore, putting

$$p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8 = S_p,$$
$$q_1+q_2+q_3+q_4+q_5+q_6+q_7+q_8 = S_q,$$

we must have $S_p + S_q = 1$.

Also, if p be the chance that the story is true, and q the chance that it is false, $\quad p+q=1$, and $p:q = S_p : S_q$;

whence $\quad p = \dfrac{S_p}{S_p+S_q} = S_p = p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8.$

Hence it is not necessary to compute q_1, q_2, &c.

5007. (By J. L. McKenzie.)—Prove that $\frac{4}{3}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdots\cdots\frac{n^2}{n^2-1}\cdots$ to ∞ (where 4, 9, 25 ... are the squares of the prime numbers) is equal to

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{ to }\infty.$$

Solution by Professor Tanner, M.A.

$$\frac{4}{3}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdots\frac{n^2}{n^2-1}\cdots=\frac{1}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\cdots\left(1-\frac{1}{n^2}\right)\cdots}$$

$$=\frac{1}{\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\cdots}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(2^r)^2}+\cdots\right\}$$

$$=\frac{1}{\left(1-\frac{1}{5^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\cdots}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(2^r)^2}+\cdots\right\}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\cdots+\frac{1}{(3^s)^2}+\cdots\right\}$$

$$=\frac{1}{\left(1-\frac{1}{5^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\cdots}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(2^r\,3^s)^2}+\cdots\right\}.$$

The bracketed expression includes all quantities of the form $\frac{1}{(2^r\,3^s)^2}$; and since n stands for any prime number, we have finally the expression

$$=\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{r^2}+\cdots,\text{ where } r \text{ is any integer.}$$

This is a particular case of the theorem

$$\frac{2^{2n}}{2^{2n}-1}\cdot\frac{3^{2n}}{3^{2n}-1}\cdots\cdots\frac{p^{2n}}{p^{2n}-1}\cdots\cdots\text{ to }\infty=1+\frac{1}{2^{2n}}+\cdots+\frac{1}{q^{2n}}+\cdots,$$

where p is any prime number, q is any number (see Boole's *Finite Differences*, 2nd edit., p. 109, foot note.)

4806. (By S. Tebay, B.A.)—From a bag containing n cards m cards are drawn and replaced r times; find the probability that not less than s different cards are drawn.

Solution by the Proposer.

The whole number of possible cases is the number of homogeneous products of mr dimensions that can be made out of n letters, no product containing any power greater than r; and the number of favourable cases is the number of homogeneous products of mr dimensions, each containing not less than s letters, and no product involving a greater power than r.

If $mr \geqq n$, s can have all values from s to n.

If $mr < n$, s can have all values from s to mr.

Let this be γ say; and let $C_t = \dfrac{\underline{|n}}{\underline{|t}\,\underline{|n-t}}$, H_t the number of homogeneous products that can be made with t letters, taken all together, with the above limitations. Then the chance required is $\dfrac{\Sigma^{\gamma}_{s}(C_t H_t)}{\Sigma^{\gamma}_{m}(C_t H_t)}$.

Let $n=10$, $m=3$, $r=4$, $s=6$; then we have the following sets of indices:—

$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$
444	4422	44211	441111	4311111	42111111
	4431	43311	432111	4221111	33111111
	4332	43221	333111	3321111	32211111
	3333	42222	332211	3222111	22221111
		33321	322221	2222211	
		33222	222222		

$t=9$	$t=10$
411111111	3111111111
321111111	2211111111
222111111	

$$\Sigma(C_3 H_3) = \frac{\underline{|10}}{\underline{|3}\,\underline{|7}}\cdot\frac{\underline{|3}}{\underline{|3}} = 120,$$

$$\Sigma(C_4 H_4) = \frac{\underline{|10}}{\underline{|4}\,\underline{|6}}\left\{\frac{\underline{|4}}{\underline{|2}\,\underline{|2}}+\frac{\underline{|4}}{\underline{|2}}+\frac{\underline{|4}}{\underline{|2}}+\frac{\underline{|4}}{\underline{|4}}\right\} = 1260,$$

$$\Sigma(C_5 H_5) = \frac{\underline{|10}}{\underline{|5}\,\underline{|5}}\left\{\frac{\underline{|5}}{\underline{|2}\,\underline{|2}}+\frac{\underline{|5}}{\underline{|2}\,\underline{|2}}+\frac{\underline{|5}}{\underline{|2}}+\frac{\underline{|5}}{\underline{|4}}+\frac{\underline{|5}}{\underline{|3}}+\frac{\underline{|5}}{\underline{|2}\,\underline{|3}}\right\} = 39060,$$

$$\Sigma(C_6 H_6) = \frac{\underline{|10}}{\underline{|6}\,\underline{|4}}\left\{\frac{\underline{|6}}{\underline{|2}\,\underline{|4}}+\frac{\underline{|6}}{\underline{|3}}+\frac{\underline{|6}}{\underline{|3}\,\underline{|3}}+\frac{\underline{|6}}{\underline{|2}\,\underline{|2}\,\underline{|2}}+\frac{\underline{|6}}{\underline{|4}}+\frac{\underline{|6}}{\underline{|6}}\right\} = 57960,$$

$$\Sigma(C_7 H_7) = \frac{\underline{|10}}{\underline{|7}\,\underline{|3}}\left\{\frac{\underline{|7}}{\underline{|5}}+\frac{\underline{|7}}{\underline{|2}\,\underline{|4}}+\frac{\underline{|7}}{\underline{|2}\,\underline{|4}}+\frac{\underline{|7}}{\underline{|3}\,\underline{|3}}+\frac{\underline{|7}}{\underline{|5}\,\underline{|2}}\right\} = 49560,$$

$$\Sigma(C_8 H_8) = \frac{\underline{|10}}{\underline{|8}\,\underline{|2}}\left\{\frac{\underline{|8}}{\underline{|6}}+\frac{\underline{|8}}{\underline{|2}\,\underline{|6}}+\frac{\underline{|8}}{\underline{|2}\,\underline{|5}}+\frac{\underline{|8}}{\underline{|4}\,\underline{|4}}\right\} = 14490,$$

$$\Sigma(C_9 H_9) = \frac{\underline{|10}}{\underline{|9}\,\underline{|1}}\left\{\frac{\underline{|9}}{\underline{|8}}+\frac{\underline{|9}}{\underline{|7}}+\frac{\underline{|9}}{\underline{|3}\,\underline{|6}}\right\} = 1650,$$

$$\Sigma(C_{10} H_{10}) = \frac{\underline{|10}}{\underline{|10}\,\underline{|0}}\left\{\frac{\underline{|10}}{\underline{|9}}+\frac{\underline{|10}}{\underline{|2}\,\underline{|8}}\right\} = 55;$$

$$\Sigma^{10}_{3}(C_t H_t) = 164155, \quad \Sigma^{10}_{6}(C_t H_t) = 123715.$$

Hence the chance $= \dfrac{123715}{164155} = \dfrac{24743}{32821}$.

Printed by C. F. Hodgson and Son, Gough Square Fleet Street, E.C.